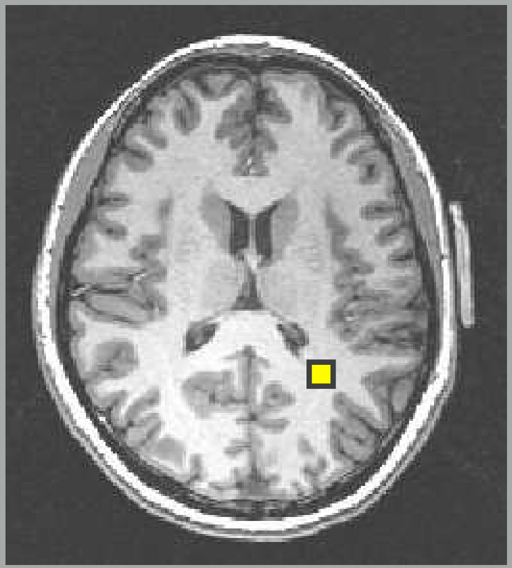
Lecture 9
Estimating the diffusion tensor

Please install AFNI
http://afni.nimh.nih.gov/afni/

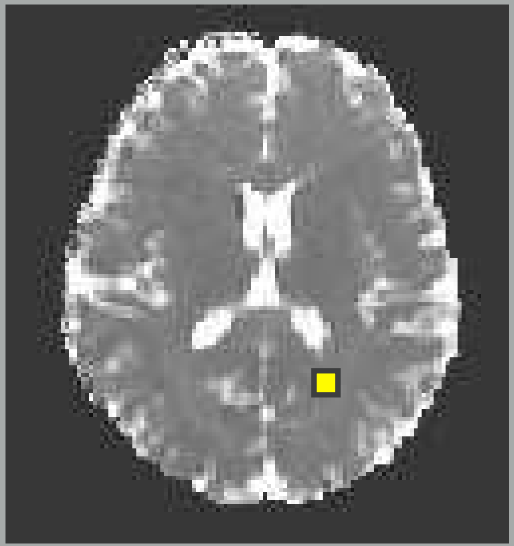
For help, contact Tom Lesperance at:
tlesperance@ucsd.edu

Next lecture, DTI

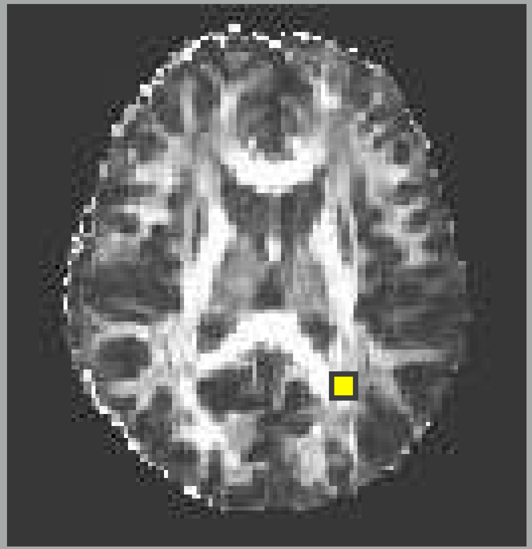
For this lecture, think in terms of a single voxel



anatomy



mean diffusivity



anisotropy

The NMR signal for 1D diffusion

$$s(q,t) = \frac{1}{\sqrt{4\pi D\tau}} \int e^{r^2/(4D\tau)}\, e^{-iqr}\, dr$$

$$\boxed{s(q,\tau) = s(0)e^{-bD}}$$

where $\quad b = q^2\tau = G^2\delta^2(\Delta - \delta/3)$

The NMR signal for 3D Gaussian diffusion

$$s(q,\tau) = \int P(\bar{r},\tau) e^{-iq\cdot\bar{r}}\, d\bar{r}$$

$$P(\bar{r},\tau) = \frac{1}{\sqrt{(4\pi\tau)^3|D|}} e^{-\bar{r}^t D^{-1} \bar{r}/4\tau}$$

$$\Downarrow$$

$$\boxed{s(q,\tau) = s(0)e^{-bD}}$$

The Multivariate Gaussian

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n |\mathbf{\Sigma}|}} \exp\left[-\frac{1}{2}(\mathbf{x} - \mu)^t \mathbf{\Sigma}^{-1}(\mathbf{x} - \mu)\right]$$

$$\mathbf{x} = \{x_1, \ldots, x_n\} \qquad \mu = \{\mu_1, \ldots, \mu_n\}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{1n} \\ \vdots & \ddots & \vdots \\ \Sigma_{n1} & \cdots & \Sigma_{nn} \end{pmatrix}$$

The Multivariate Gaussian in 2 Variables

$$\mathbf{\Sigma} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

$$\mathbf{x} = \{x, y\}$$

$$\mu_{\mathbf{x}} = \{\mu_x, \mu_y\}$$

The matrix inverse

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$M = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \qquad M^{-1} = \frac{1}{ab - c^2} \begin{pmatrix} b & -c \\ -c & a \end{pmatrix}$$

$$M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad M^{-1} = \frac{1}{ab} \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}$$

The Exponent of the Multivariate Gaussian

$$\begin{pmatrix} x - \mu_x \\ y - \mu_y \end{pmatrix}^t \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} x - \mu_x \\ y - \mu_y \end{pmatrix}$$

$$\frac{(x - \mu_x)^2}{a} + \frac{(y - \mu_y)^2}{b}$$

If the matrix M is diagonal, we get a simpler form

The Multivariate Gaussian

$$p(x,y) = \frac{1}{\sqrt{(2\pi)^2 ab}} \exp - \frac{1}{2} \left[\frac{(x-\mu_x)^2}{a} + \frac{(y-\mu_y)^2}{b} \right]$$

$$= \frac{1}{\sqrt{2\pi a}} \exp - \frac{1}{2} \left[\frac{(x-\mu_x)^2}{a} \right] \frac{1}{\sqrt{2\pi b}} \exp - \frac{1}{2} \left[\frac{(y-\mu_y)^2}{b} \right]$$

$$= p(x)p(y)$$

Estimating the Diffusion Coefficient

matlab program fit1D.m

Estimating the Diffusion Tensor

matlab program fit2D.m

Estimating the Diffusion Tensor

In practice

D calculated with 3dDWItoDT (AFNI)

The diffusion ellipsoid

1. Estimate D from signal: $$s(q,\tau) = s(0)e^{-bD}$$

2. Use D to reconstruct contour of equal probability: $$P(\bar{r},\tau) = \frac{1}{\sqrt{(4\pi\tau)^3|D|}}e^{-\bar{r}^t D^{-1}\bar{r}/4\tau}$$

$$P(\bar{r},\tau) = \mathrm{C}e^{-\frac{1}{2}\bar{r}^t A\bar{r}} \qquad A = \frac{D^{-1}}{2\tau}$$

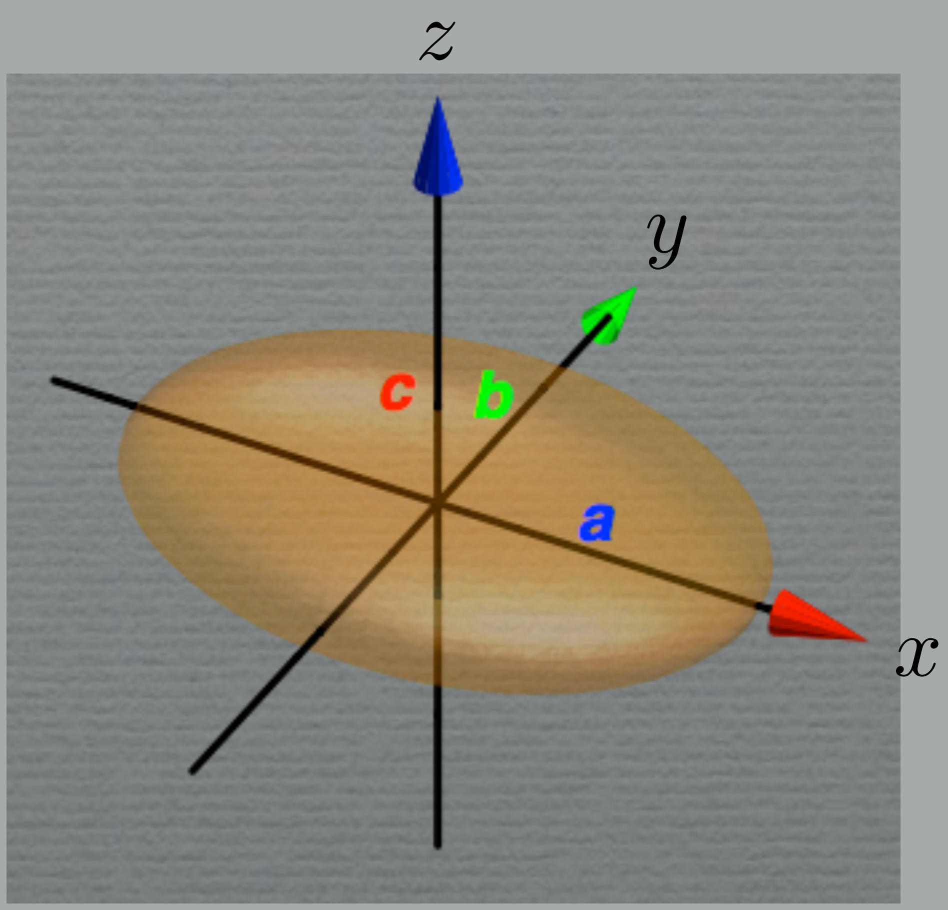
Inverse covariance matrix

The diffusion ellipsoid

$$P(\bar{r}, \tau) = \mathrm{C}e^{-\frac{1}{2}\bar{r}^t A \bar{r}} \qquad A = \frac{D^{-1}}{2\tau}$$

$$\bar{r}^t A \bar{r} = 1 \quad \Rightarrow \quad \frac{\bar{r}^t D^{-1} \bar{r}}{2\tau} = 1$$

An ellipsoid



$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

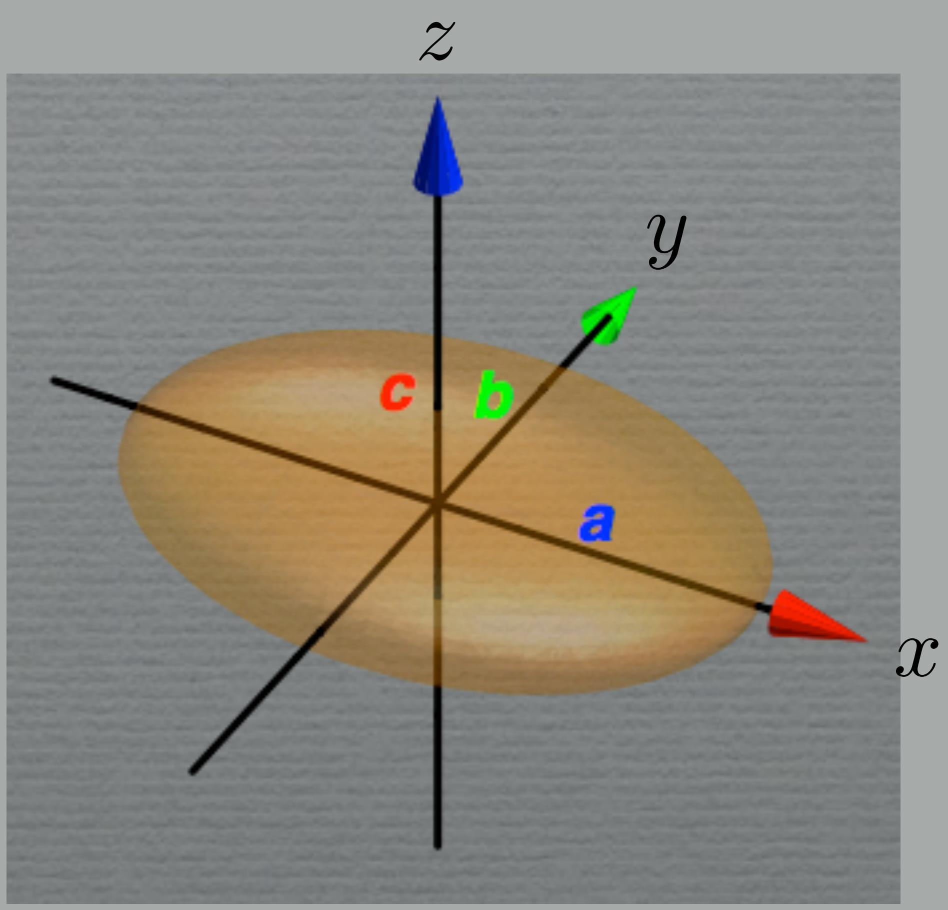
An ellipsoid in matrix form

$$f(\boldsymbol{x}) = \boldsymbol{x}^t \boldsymbol{\Lambda} \boldsymbol{x} \qquad \boldsymbol{x} = \{x, y, z\}$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_z \end{pmatrix}$$

$$\lambda_x = \frac{1}{a^2} \qquad \lambda_y = \frac{1}{b^2} \qquad \lambda_z = \frac{1}{c^2}$$

An ellipsoid in matrix form



distance along axes from origin to ellipsoid

$$a = \frac{1}{\sqrt{\lambda_x}} \qquad b = \frac{1}{\sqrt{\lambda_y}} \qquad c = \frac{1}{\sqrt{\lambda_z}}$$

Diffusion Tensor (aligned along x,y,z)

$$\mathbf{D} = \begin{pmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_z \end{pmatrix}$$ "diagonal matrix"

where

$$\begin{aligned} \lambda_x &= D_{xx} \\ \lambda_y &= D_{yy} \\ \lambda_z &= D_{zz} \end{aligned}$$

Diffusion ellipsoid

$$\frac{\bar{r}^t D^{-1} \bar{r}}{2\tau} = 1 \qquad r = \{x', y', z'\}$$

$$\left(\frac{x'}{\sqrt{2\lambda_x \tau}}\right)^2 + \left(\frac{y'}{\sqrt{2\lambda_y \tau}}\right)^2 + \left(\frac{z'}{\sqrt{2\lambda_z \tau}}\right)^2 = 1$$

$$\lambda_i = \text{principal diffusivities}$$

But, from last lecture, diffusion tensor looks like this...

$$\boldsymbol{D} = \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{xy} & D_{yy} & D_{yz} \\ D_{xz} & D_{yz} & D_{zz} \end{pmatrix}$$

not diagonal

However, we do know that *D* is real and symmetric

$$\begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix} = \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{xy} & D_{yy} & D_{yz} \\ D_{xz} & D_{yz} & D_{zz} \end{pmatrix}$$

$$\boldsymbol{D} = \boldsymbol{D}^t$$ matrix form

$$D_{ij} = D_{ji}$$ component form

Spectral Theorem

A real, symmetric matrix can be diagonalized

Importantly, the inverse of a symmetric matrix is also symmetric

Diagonalizing the Diffusion Tensor

$$\boldsymbol{D} = \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix}$$

$$\boldsymbol{D}_\Lambda = \boldsymbol{R}^t \boldsymbol{D} \boldsymbol{R}$$

$$\boldsymbol{D}_\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$\lambda_i =$ principal diffusivities

Diagonalizing the Diffusion Tensor

$$r^t D^{-1} r$$

$$r^t (R D_\Lambda^{-1} R^t) r$$

D_Λ^{-1} is diagonal

$$(r^t R) D_\Lambda^{-1} (R^t r)$$

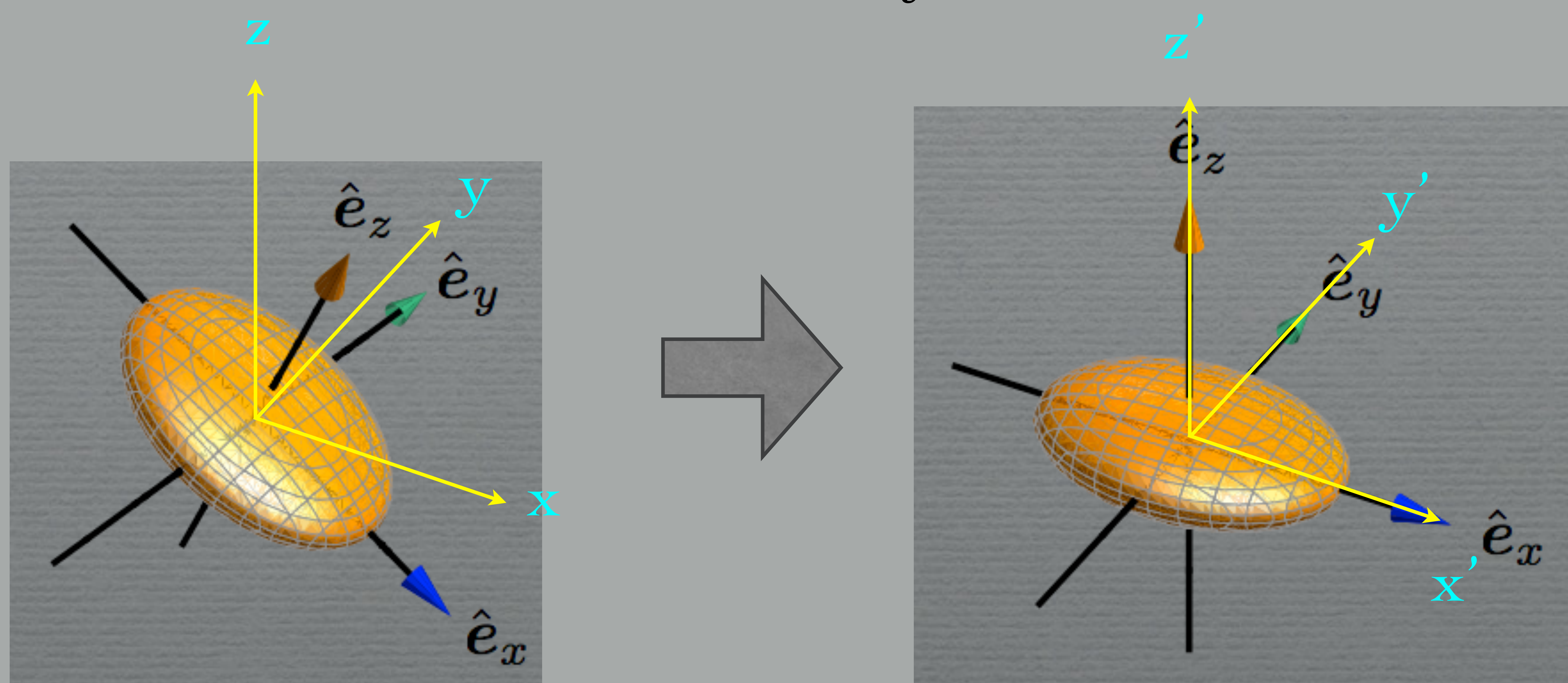
$$e^t D_\Lambda^{-1} e$$

$e = R^t r$

eigenvectors

Diagonalizing the Diffusion Tensor

Geometrically



Diagonalizing the Diffusion Tensor

These eigenvectors *define* the coordinate system of the ellipsoid and thus tell us its orientation

The eigenvalues and eigenvectors are *invariant* to rotations of the tensor

Diffusion Tensor Eigensystem

$$\boldsymbol{D}_\Lambda = \begin{pmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_z \end{pmatrix}.$$

where

$$\begin{aligned} \lambda_x &= D_{xx} \\ \lambda_y &= D_{yy} \\ \lambda_z &= D_{zz} \end{aligned}$$

The eigenvalues are the diffusion coefficients along the principal axes

Eigenvalues and Eigenvectors

An *eigenvalue equation*

$$\boldsymbol{A} \cdot \hat{\boldsymbol{e}} = \boldsymbol{\lambda} \hat{\boldsymbol{e}}$$

matrix

vector

constant

same vector

Trace of a matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{pmatrix}$$

$$\mathrm{Tr}(\boldsymbol{A}) = \sum_{i}^{n} a_{ii}$$

Trace of the Diffusion Tensor

$$\boldsymbol{D}_\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

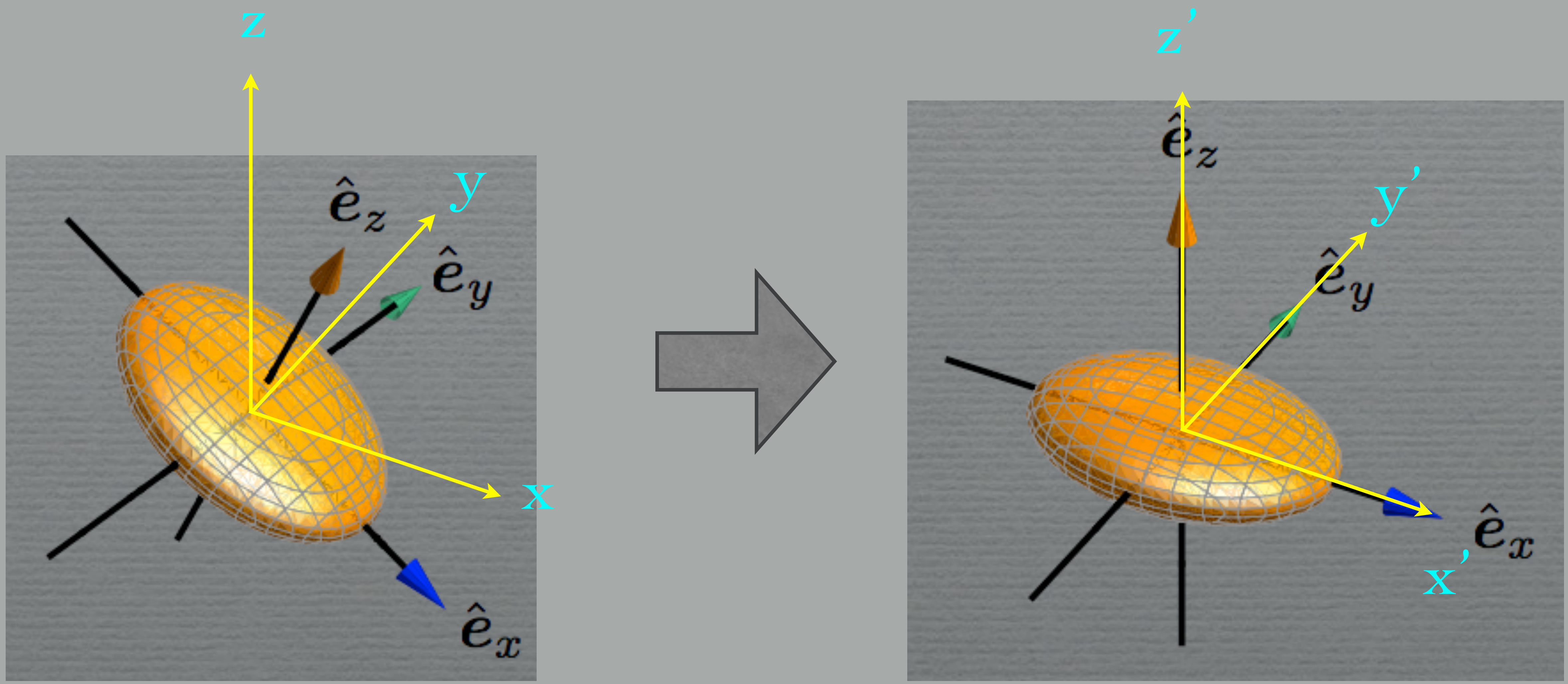
$$\mathrm{Tr}(\boldsymbol{D}_\Lambda) = \lambda_1 + \lambda_2 + \lambda_3 = \underbrace{D_{xx} + D_{yy} + D_{zz}}_{3\langle D \rangle}$$

$$\therefore \boxed{\langle D \rangle = \frac{1}{3}\mathrm{Tr}(\boldsymbol{D}_\Lambda)}$$

Remarkable property of trace

$$\therefore \quad \boxed{\mathrm{Tr}(\boldsymbol{D}) = \mathrm{Tr}(\boldsymbol{D}_\Lambda)}$$

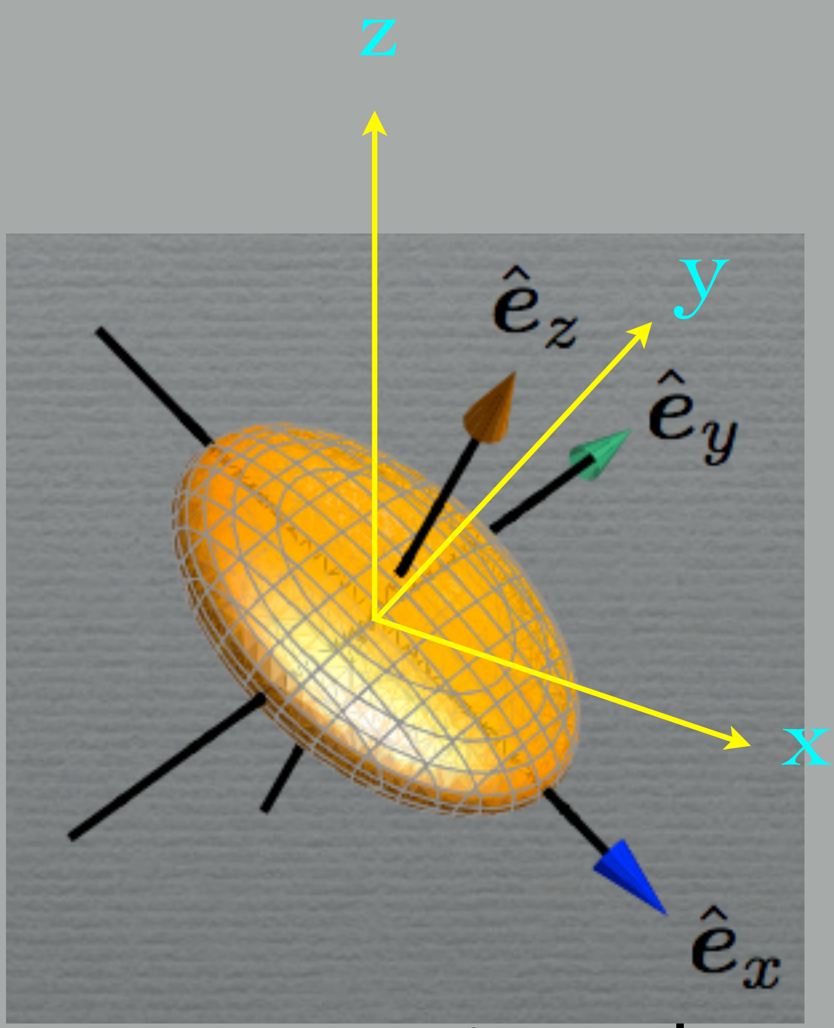
The trace is *invariant* to rotations of the tensor

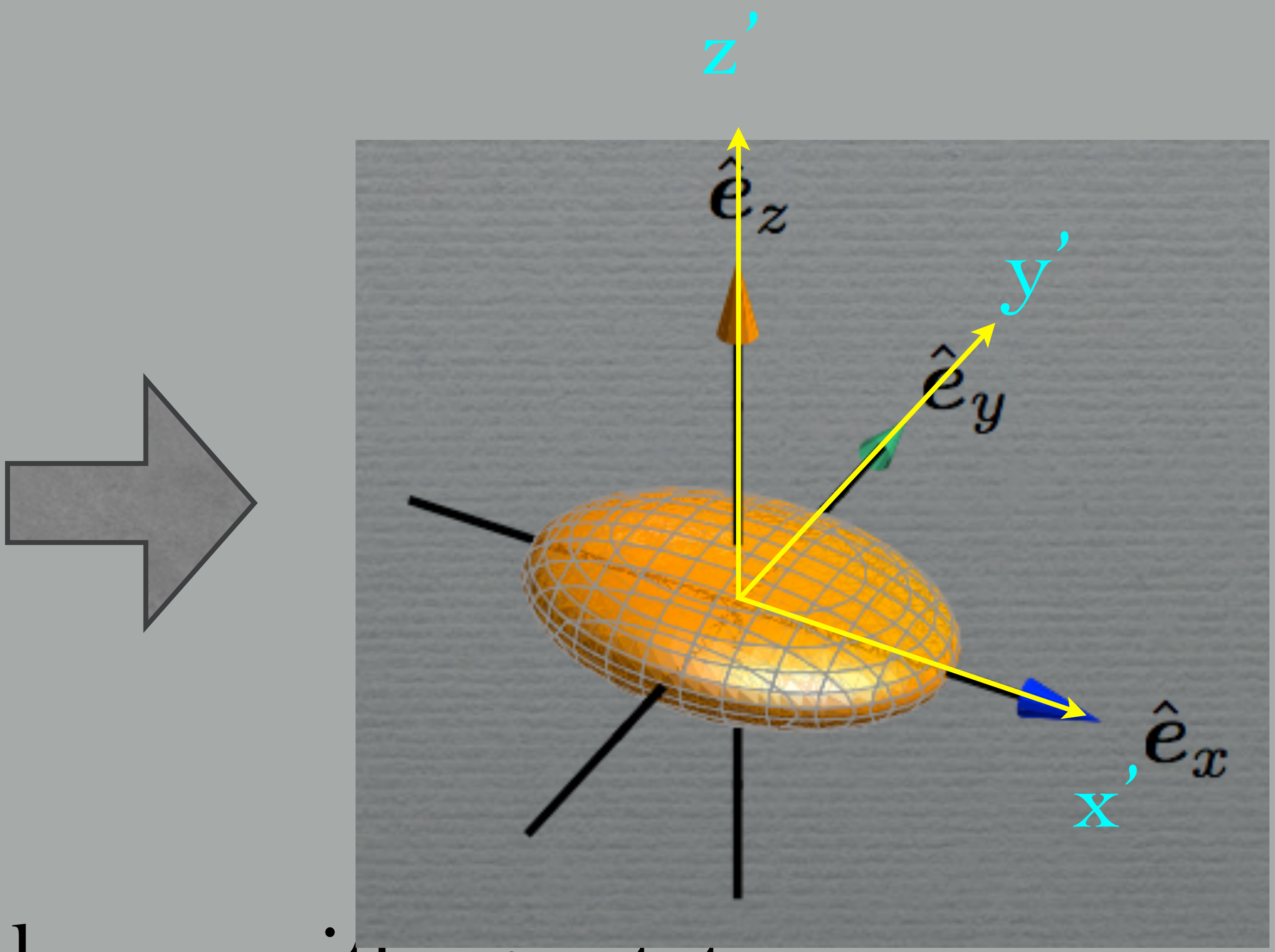


Trace of the Diffusion Tensor

$$\therefore \quad \boxed{\langle D \rangle = \frac{1}{3}\mathrm{Tr}(\boldsymbol{D})}$$

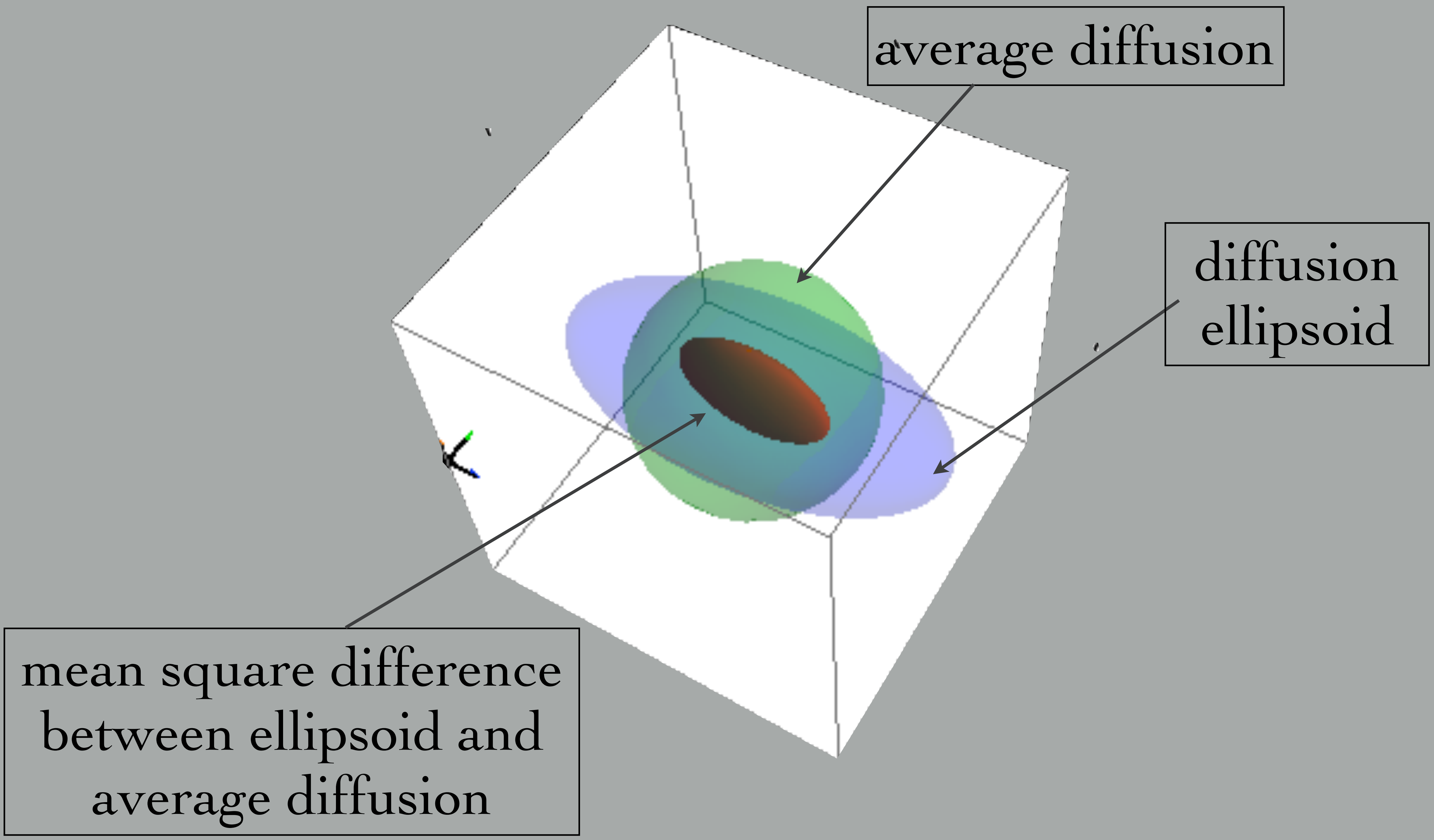
The average is *invariant* to rotations of the tensor



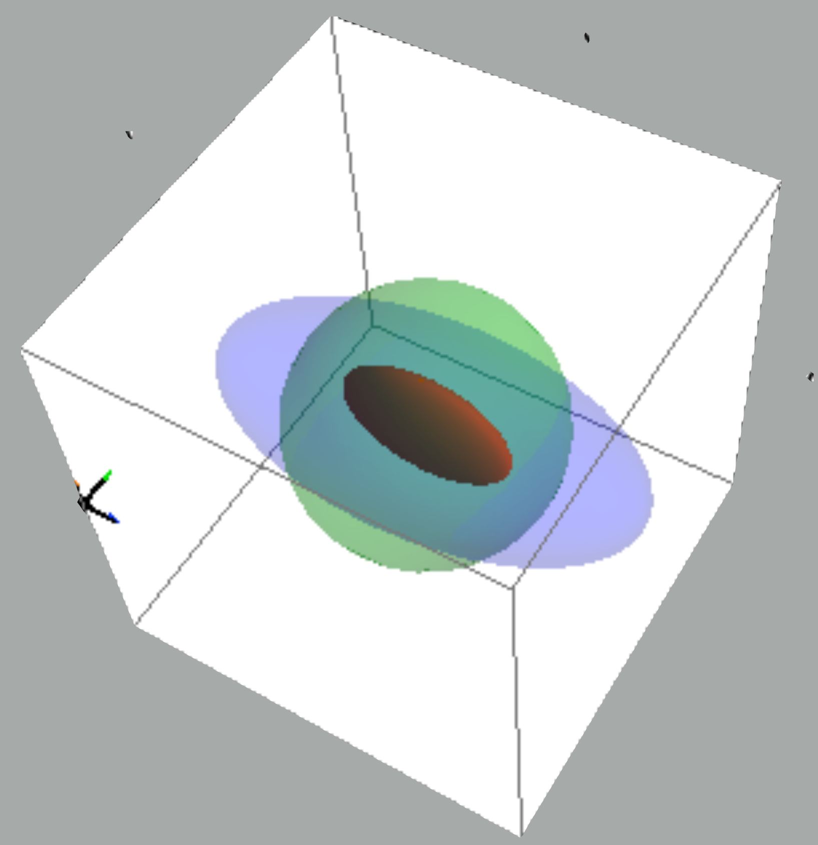


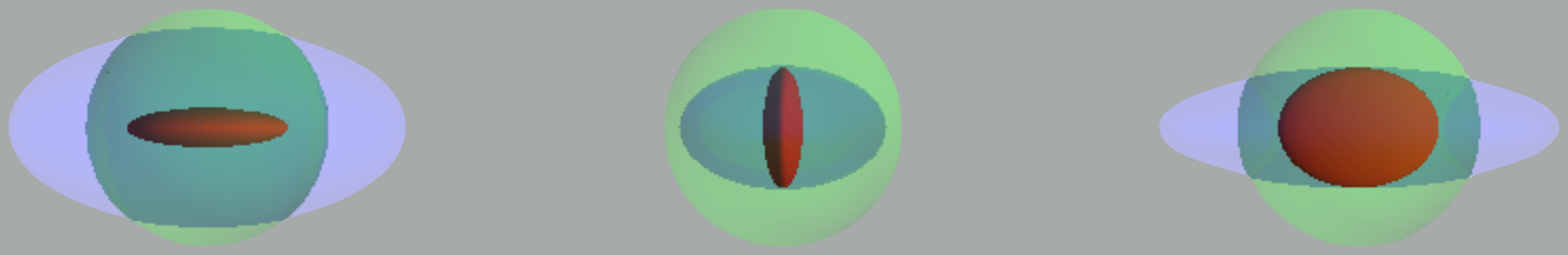
average doesn't change with orientation

Geometric Picture of the Diffusion Average and Anisotropy



Geometric Picture of the Diffusion Average and Anisotropy





Diffusion Anisotropy

Decompose D into mean and variations

$$\boldsymbol{D}_\Lambda = \begin{pmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_z \end{pmatrix} = \begin{pmatrix} \bar{\lambda} + \delta\lambda_x & 0 & 0 \\ 0 & \bar{\lambda} + \delta\lambda_y & 0 \\ 0 & 0 & \bar{\lambda} + \delta\lambda_z \end{pmatrix}$$

$$\delta\lambda_i = \lambda_i - \bar{\lambda} \quad , \quad i = \{x, y, z\}$$

$$\boldsymbol{D}_\Lambda = \underbrace{\begin{pmatrix} \bar{\lambda} & 0 & 0 \\ 0 & \bar{\lambda} & 0 \\ 0 & 0 & \bar{\lambda} \end{pmatrix}}_{\bar{\boldsymbol{D}}} + \underbrace{\begin{pmatrix} \delta\lambda_x & 0 & 0 \\ 0 & \delta\lambda_y & 0 \\ 0 & 0 & \delta\lambda_z \end{pmatrix}}_{\delta\boldsymbol{D}}$$

Diffusion Anisotropy

variations

$$\delta \boldsymbol{D} = \begin{pmatrix} \delta\lambda_1 & 0 & 0 \\ 0 & \delta\lambda_2 & 0 \\ 0 & 0 & \delta\lambda_3 \end{pmatrix}$$

$$\sqrt{\overline{(\delta\lambda)^2}} = \sqrt{\frac{1}{3}\sum_{i=1}^{3}(\delta\lambda_i)^2} = \sqrt{\frac{1}{3}\delta\boldsymbol{D}^t\delta\boldsymbol{D}}$$

$$= \frac{1}{\sqrt{3}}\sqrt{(\lambda_1-\bar{\lambda})^2+(\lambda_2-\bar{\lambda})^2+(\lambda_3-\bar{\lambda})^2}$$

$$= \frac{1}{\sqrt{3}}\sqrt{\sigma_\lambda^2}$$

Diffusion Anisotropy

$$\boldsymbol{D}_\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\sqrt{\overline{\lambda^2}} = \sqrt{\frac{1}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)} = \sqrt{\frac{1}{3}\sum_i \lambda_i^2} = \sqrt{\frac{1}{3}\boldsymbol{D}_\Lambda^t \boldsymbol{D}_\Lambda}$$

Diffusion Anisotropy

So a natural measure of anisotropy is variance over mean squared eigenvalues

$$\sqrt{\frac{\overline{(\delta\lambda)^2}}{\overline{\lambda^2}}}$$

Diffusion Anisotropy

But notice that if

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{1, 0, 0\}$$

$$\overline{\lambda^2} = \frac{1}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = \frac{1}{3}$$

$$\overline{(\delta\lambda)^2} = \frac{1}{3}[(\lambda_1 - \bar{\lambda})^2 + (\lambda_2 - \bar{\lambda})^2 + (\lambda_3 - \bar{\lambda})^2] = \frac{1}{3}\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = \frac{2}{9}$$

$$\sqrt{\frac{\overline{(\delta\lambda)^2}}{\overline{\lambda^2}}} = \sqrt{\frac{2/9}{1/3}} = \sqrt{\frac{2}{3}}$$

Diffusion Anisotropy

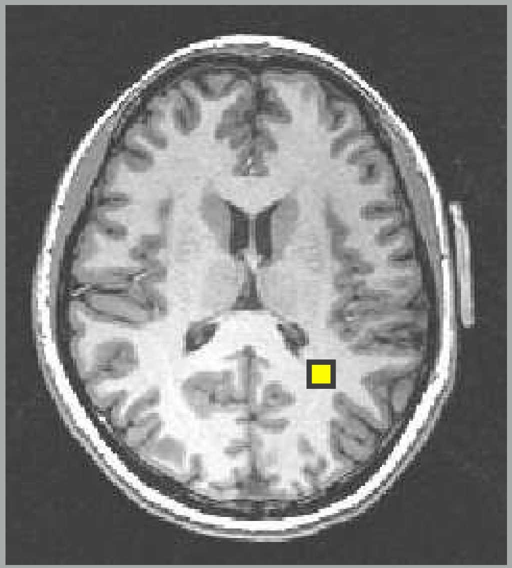
Normalize so that max value = 1
by multiplying by $\sqrt{3/2}$
and define

Fractional Anisotropy

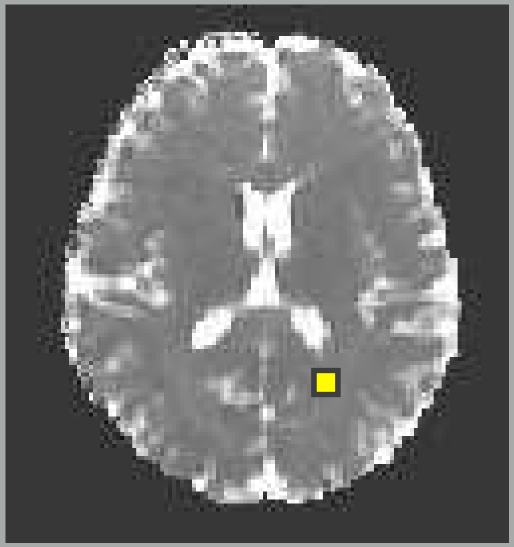
$$FA = \sqrt{\frac{3}{2}\frac{\overline{(\delta\lambda)^2}}{\overline{\lambda^2}}}$$

Next lecture, DTI

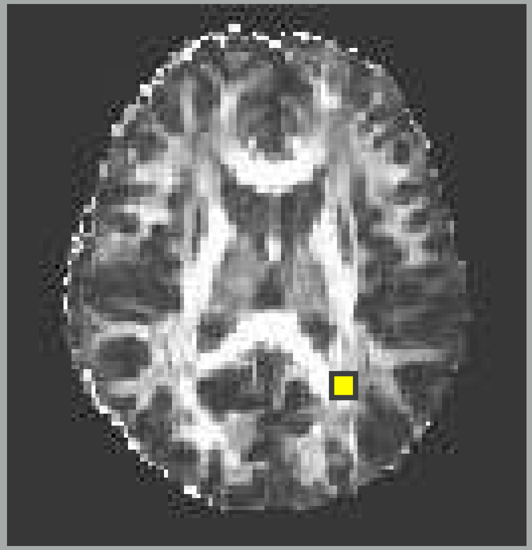
For this lecture, think in terms of a single voxel



anatomy



mean diffusivity



anisotropy