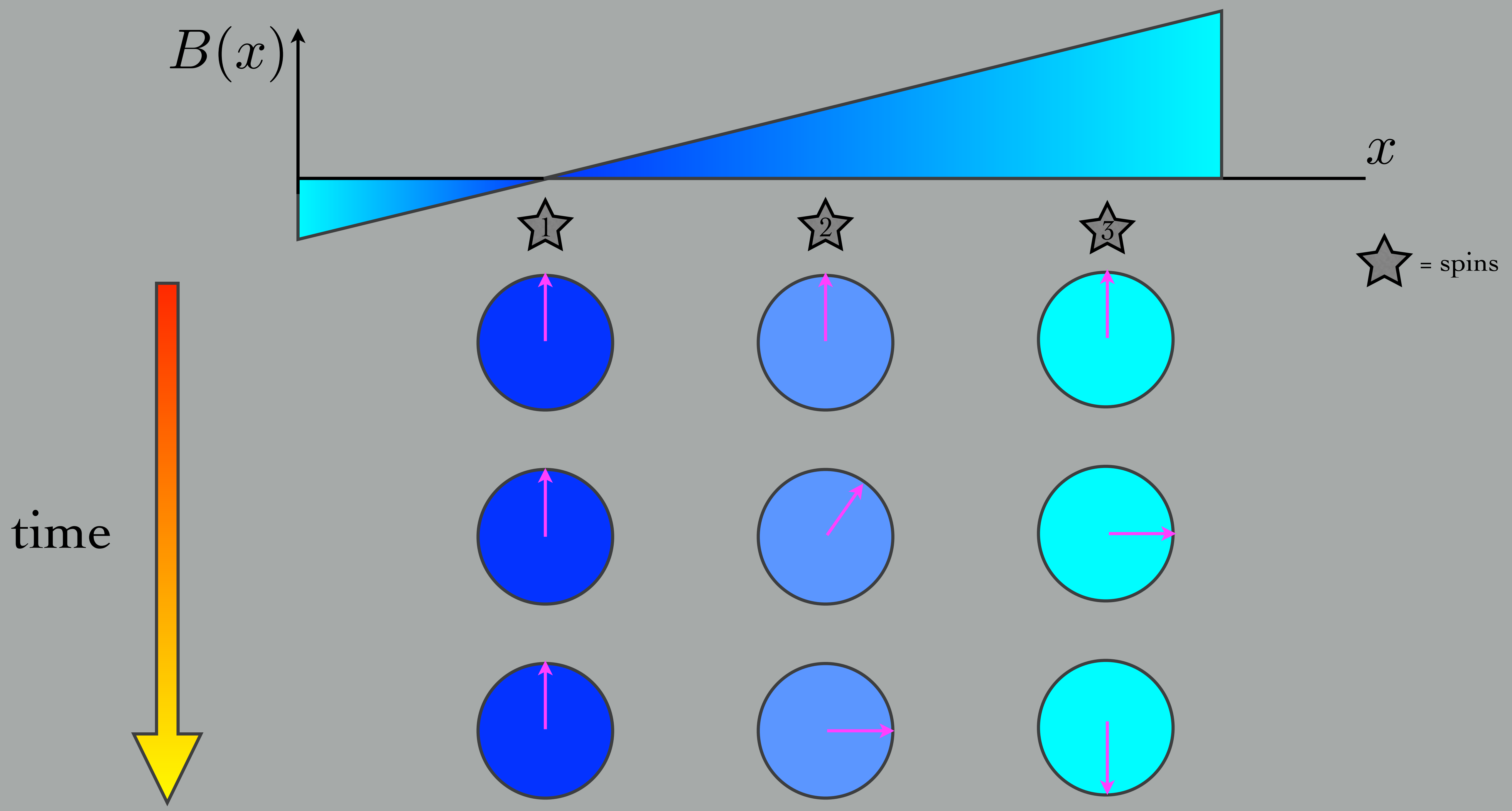
Lecture 7

The MR Sensitivity to Diffusion

The Basic Ingredient #1:

Precession phase in gradient G_x



stationary isochromats

The NMR Signal

$$s(\omega) = \int_{\Omega} m_{\perp}(\boldsymbol{r}, t) e^{-i\varphi(x,\tau)}\, d\boldsymbol{r}$$

where $$\varphi(x, \tau) = \int_0^{\tau} G(t) x(t)\, dt$$

The signal is the *Fourier Transform* of the transverse magnetization

The Phase of Moving Spins

$$\varphi(\tau) = \int_0^\tau G(t)x(t)\,dt$$

The phase depends upon:

1. Temporal profile of the gradient $G(t)$
2. The motions $x(t)$

Some notation

$$x_o^{(n)} \equiv \frac{d^n x}{dt}\Big|_{t=0}$$

the n'th order time derivative of x
evaluate at time $t=0$

Description of motion

$$x(t) = x_o + x_o^{(1)} t + \frac{1}{2!} x_o^{(2)} t^2 + \ldots + \frac{1}{n!} x_o^{(n)} t^n$$

$$x_o^{(0)} \equiv x(0)$$ initial position

$$x_o^{(1)} \equiv \frac{dx}{dt}|_{t=0} = v(0)$$ initial velocity

$$x_o^{(2)} \equiv \frac{d^2x}{dt}|_{t=0} = a(0)$$ initial acceleration

(etc.)

The Phase of Moving Spins

$$\varphi(\tau) = \int_0^\tau G(t) x(t) \, dt$$

$$\boxed{\frac{1}{n!} x_o^{(n)} t^n}$$

$$\varphi_n(\tau) = \frac{1}{n!} x_o^{(n)} \underbrace{\int_0^\tau G(t) t^n \, dt}_{m_n(\tau)}$$

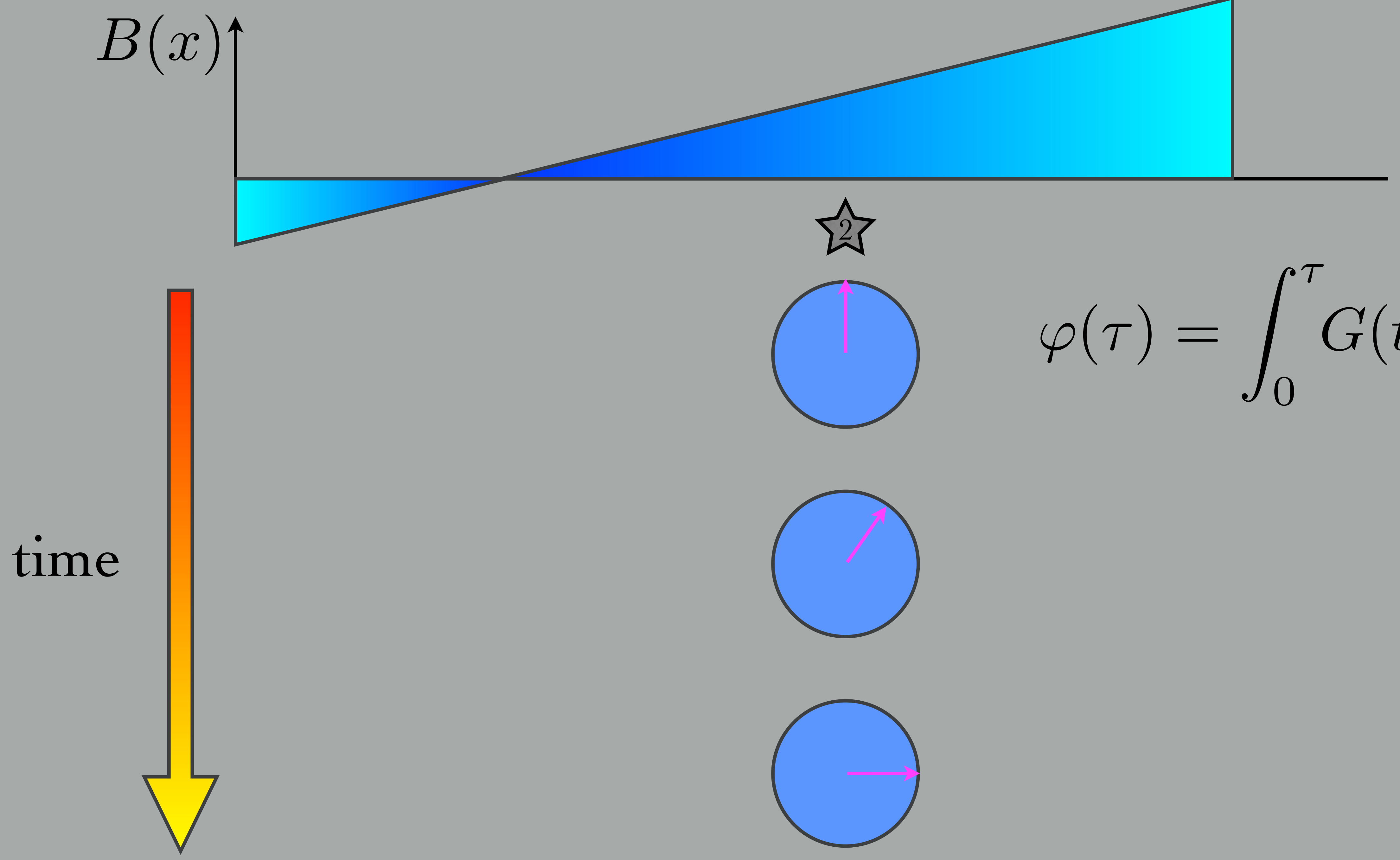
The Phase of Moving Spins

$$\varphi_n(\tau) = \frac{1}{n!} x_o^{(n)} m_n(\tau)$$

phase contribution of n'th order of motion	=	n'th time derivative of displacement	n'th gradient moment

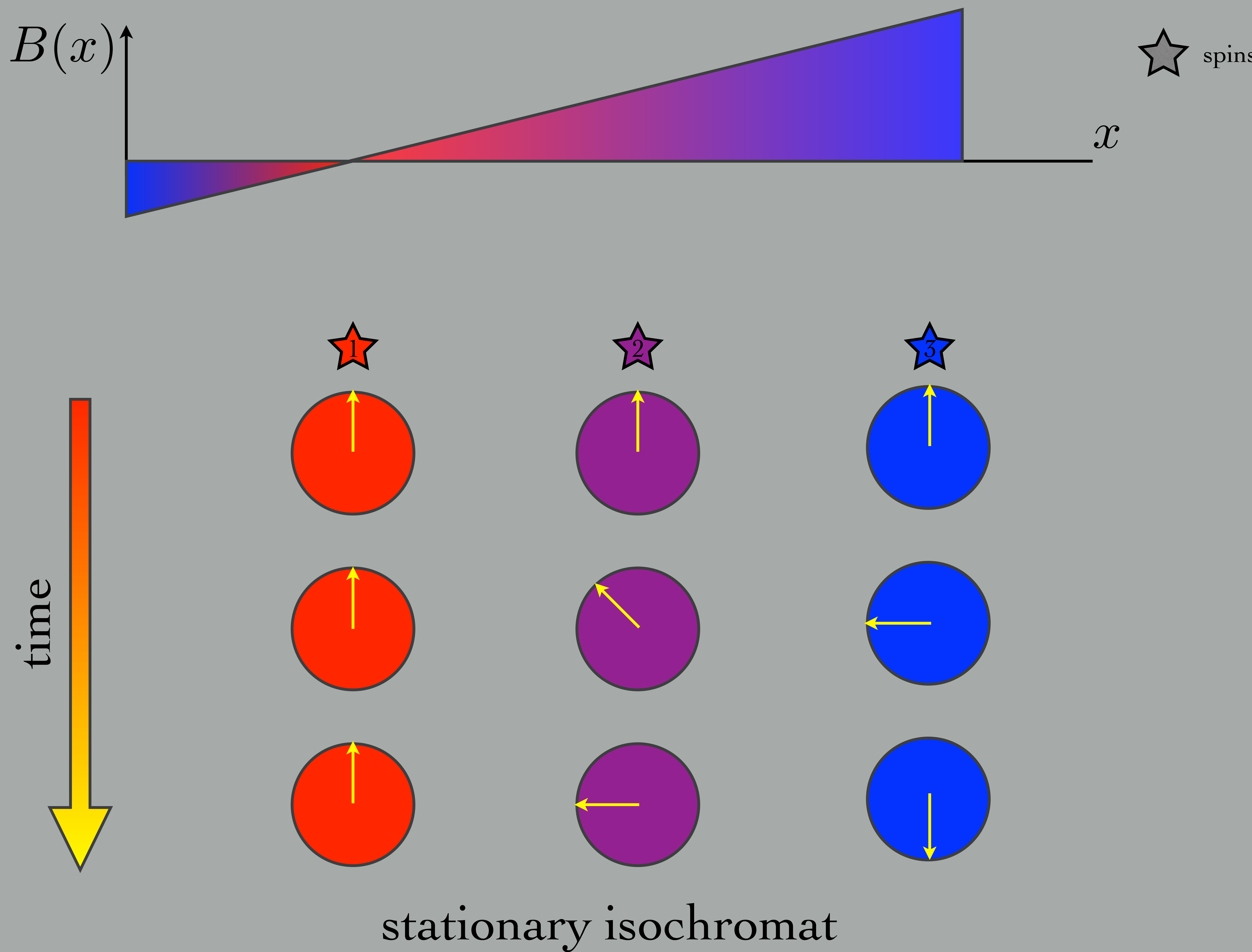
$$m_n(\tau) = \int_0^\tau G(t) t^n \, dt$$

The Phase of Moving Spins

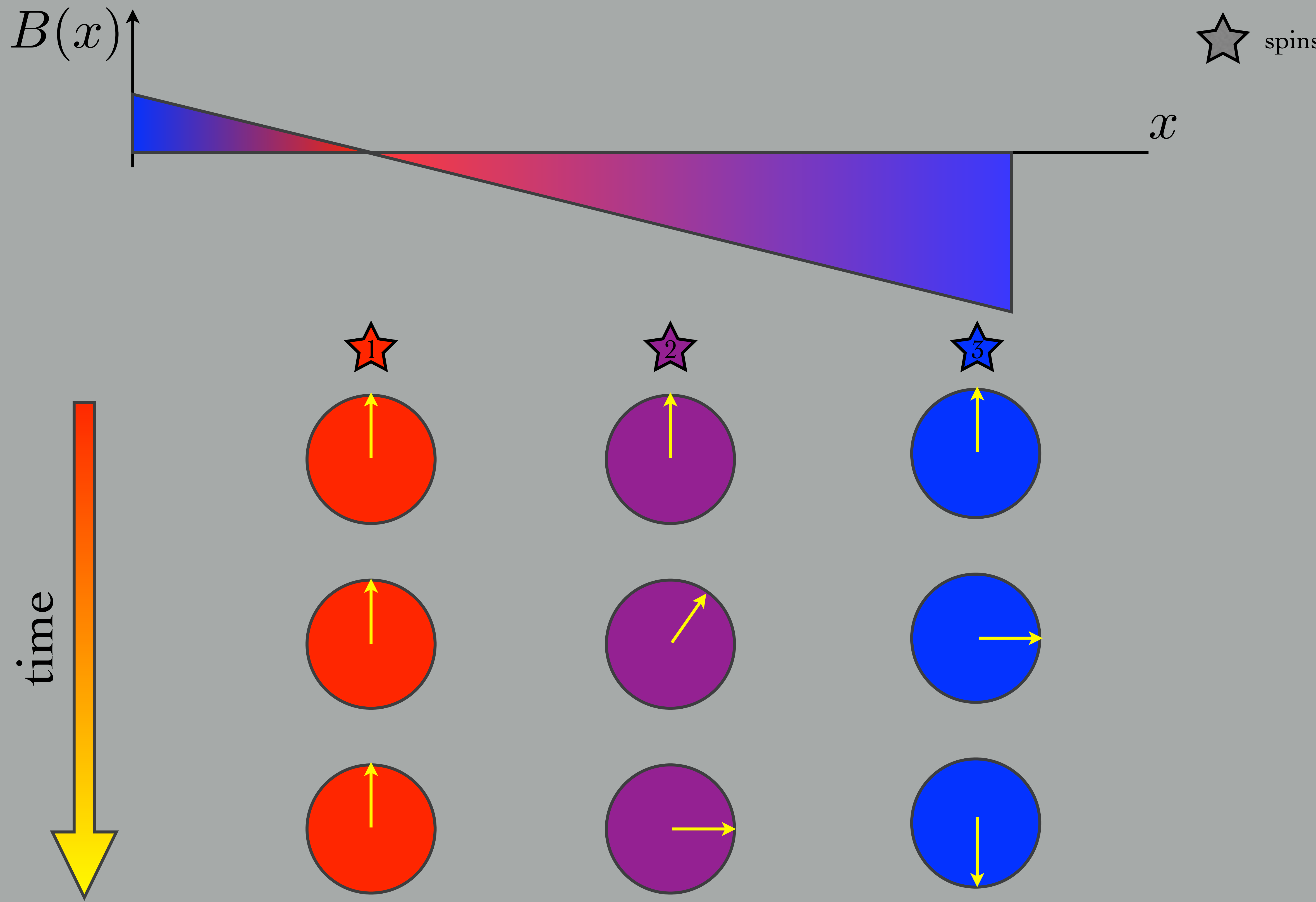


$$\varphi(\tau) = \int_0^\tau G(t)x(t)\,dt$$

Precession phase in a positive gradient

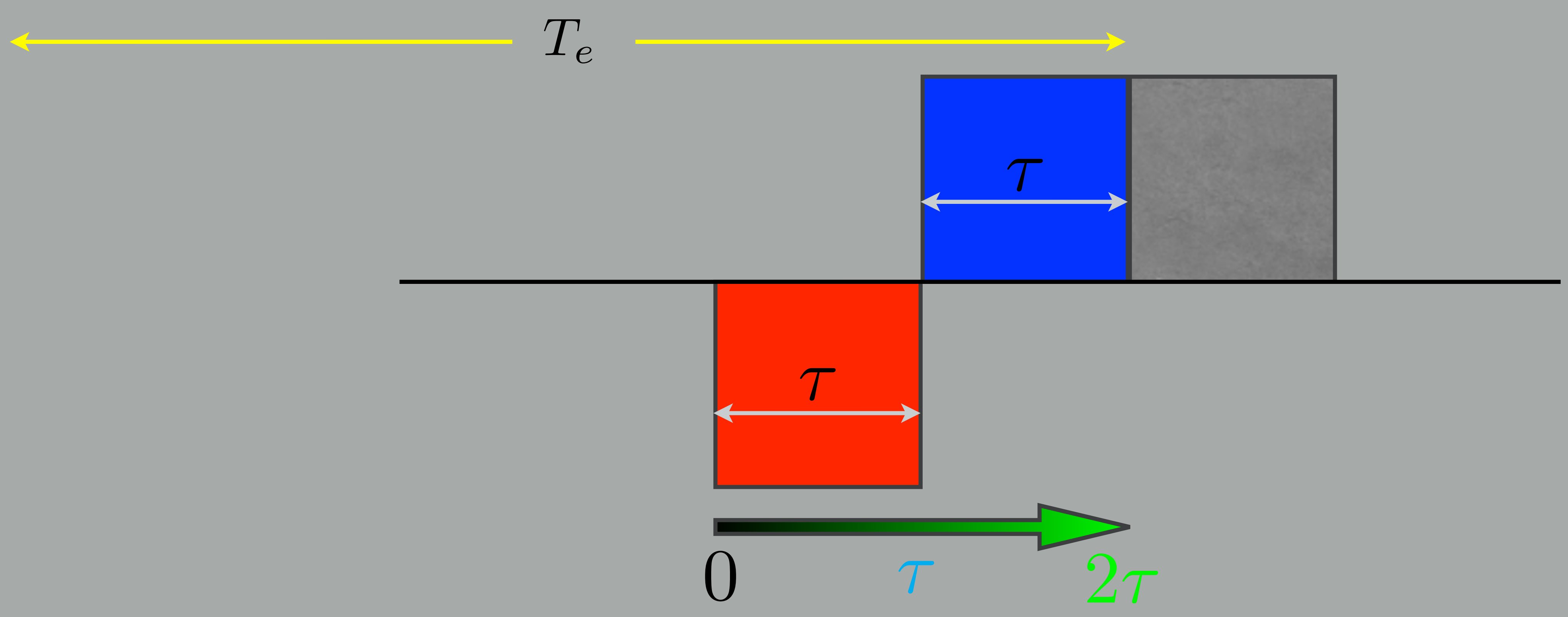


Precession phase in a positive gradient



stationary isochromat

Gradient echo first moment

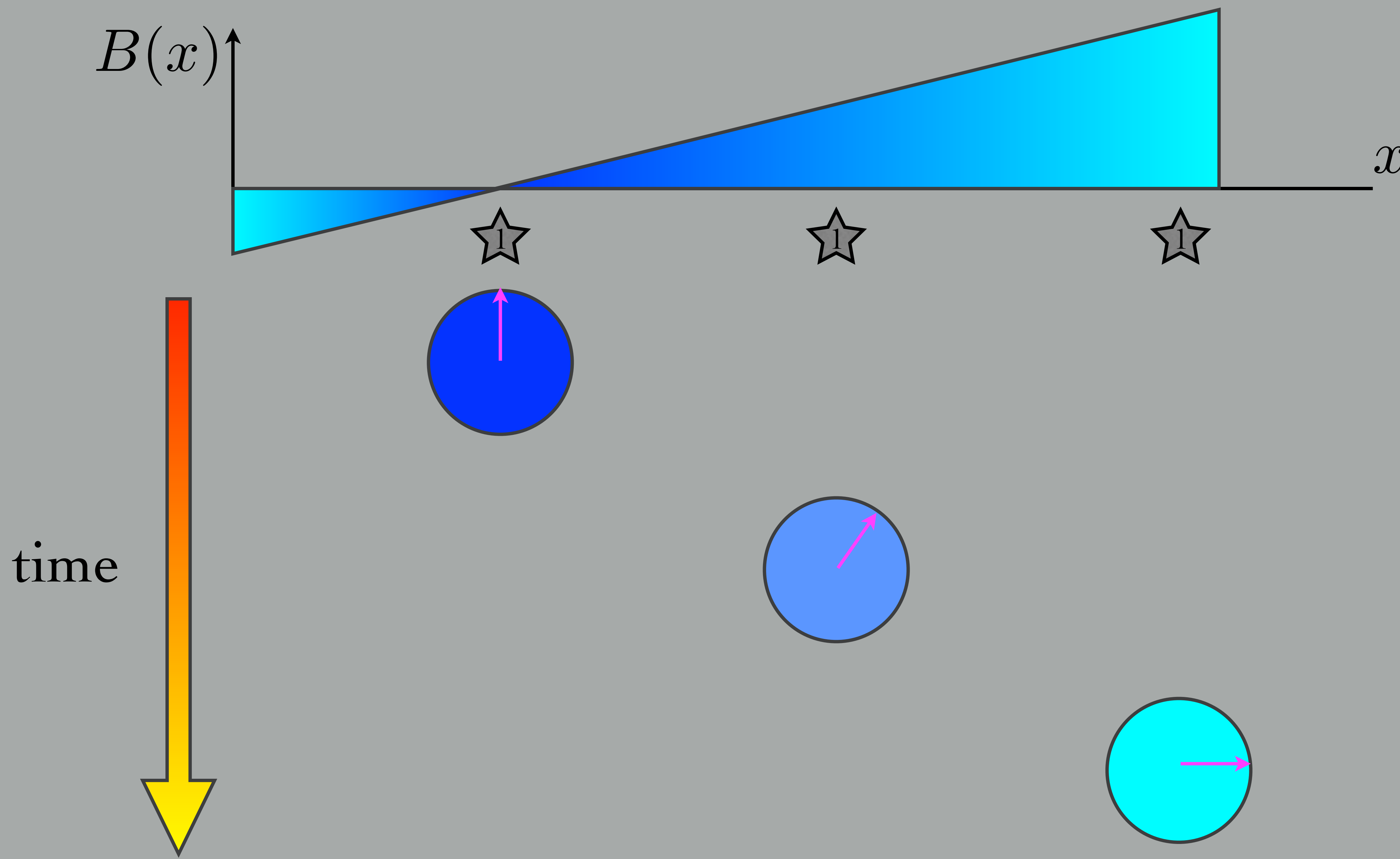


$$m_0(T_e) = \int_0^{T_e} G(t) \; dt = G\int_0^{\tau} dt - G\int_{\tau}^{2\tau} dt$$

$$= 0$$

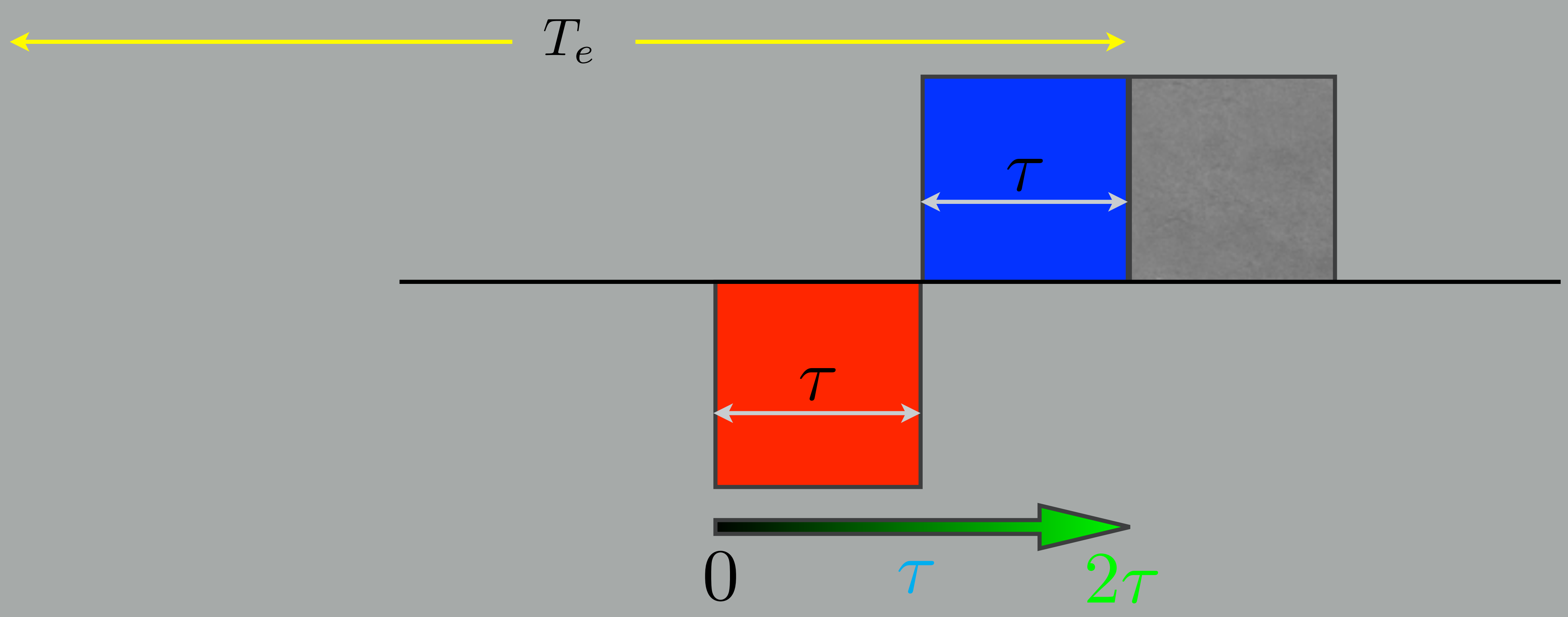
The Basic Ingredient #1:

Precession phase in a gradient



isochromat moving at velocity v

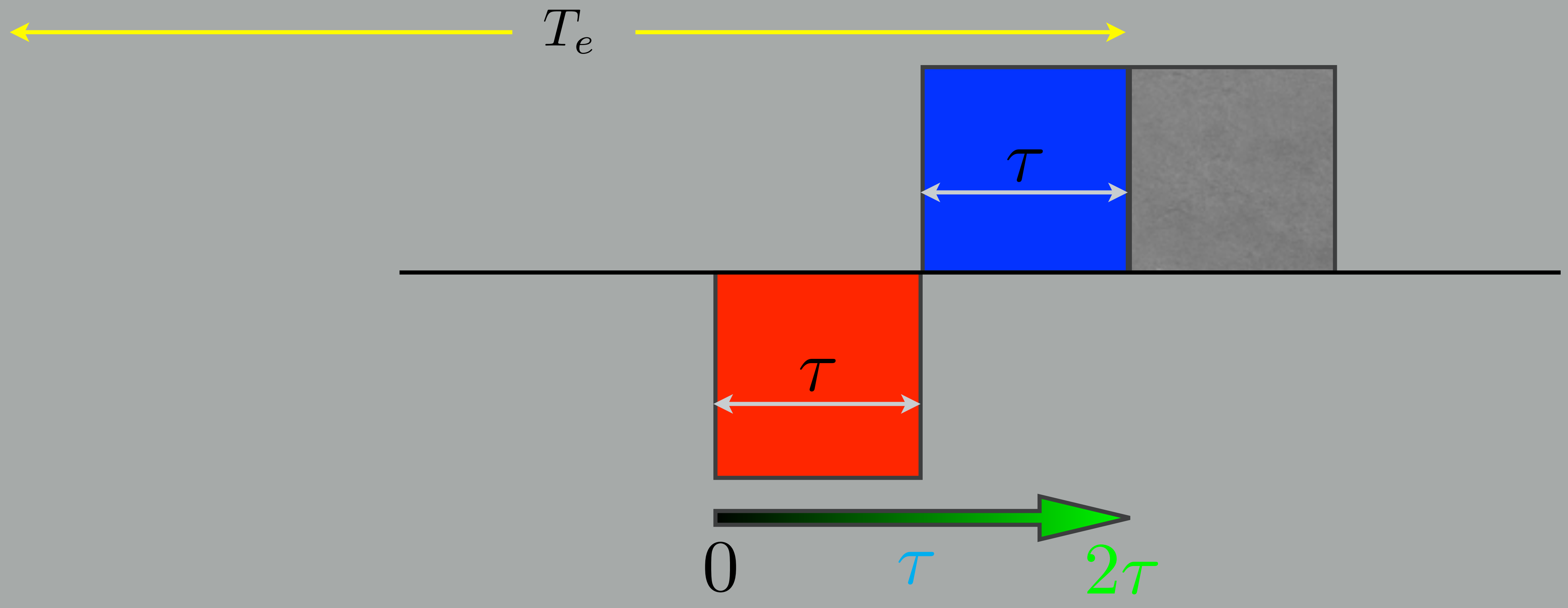
Gradient echo second moment



$$m_1(T_e) = \int_0^{T_e} G(t)t\,dt = G\int_0^{\tau} t\,dt - G\int_{\tau}^{2\tau} t\,dt$$

$$= G\tau^2$$

Gradient echo phases

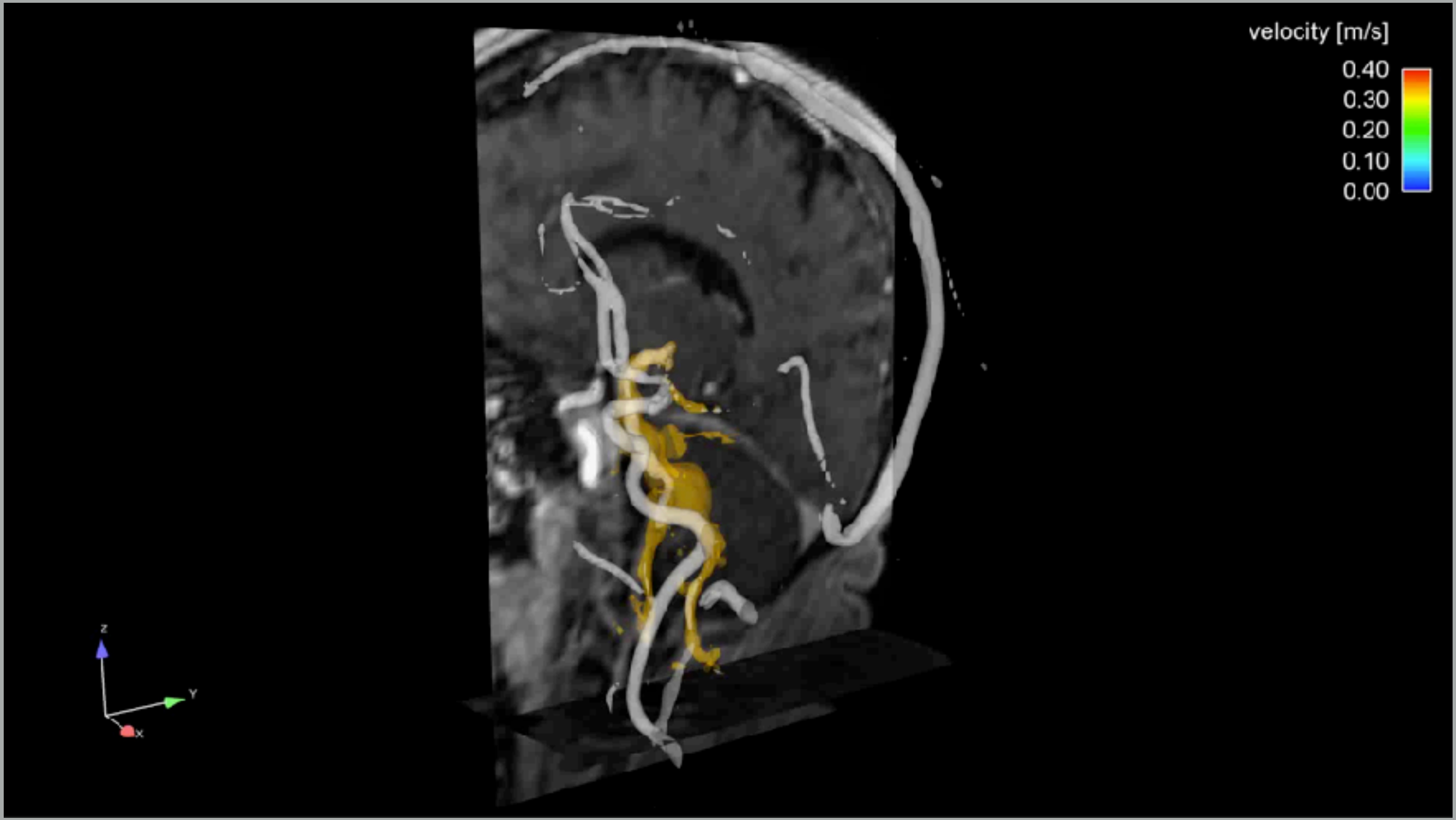


$$\varphi_o(T_e) = \quad x_o m_o(T_e) = 0$$

$$\varphi_1(T_e) = \frac{1}{2} x_1 m_1(T_e) = \frac{1}{2} G v \tau^2$$

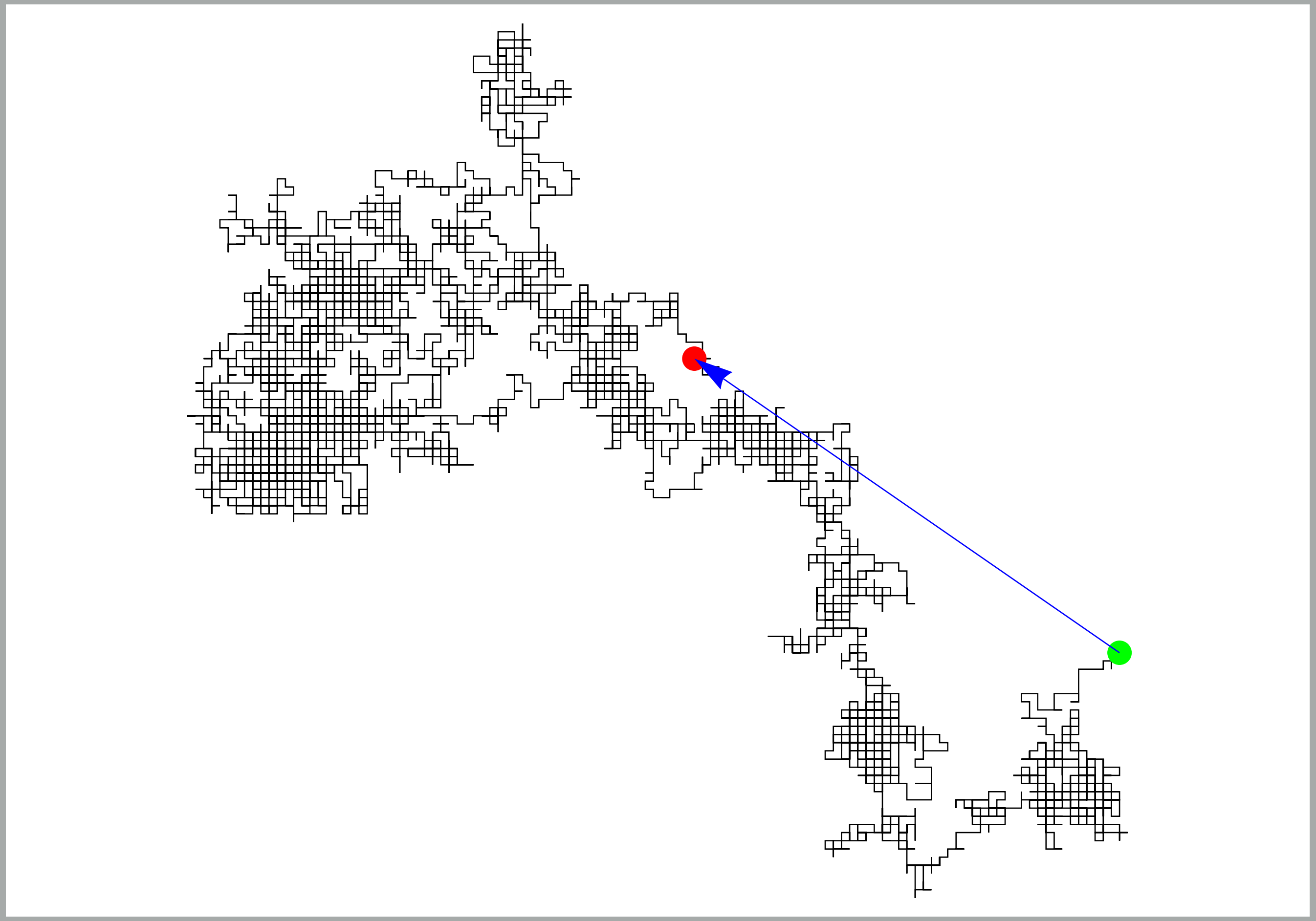
flow induced phase!

Magnetic Resonance Phase Contrast Angiography



Magnetic Resonance Angiography

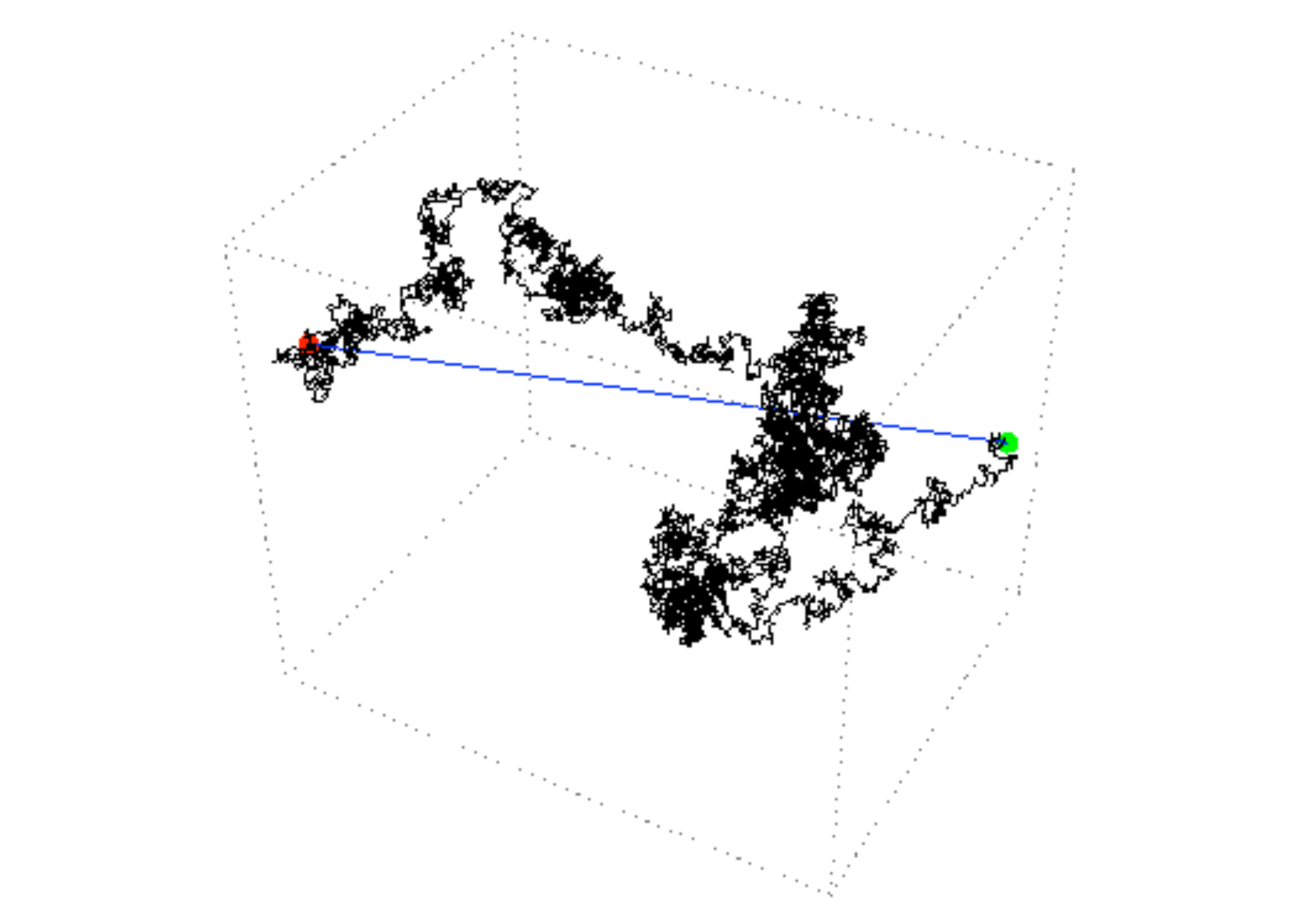
The Random Walk



start
end

Two-dimensional

The Random Walk



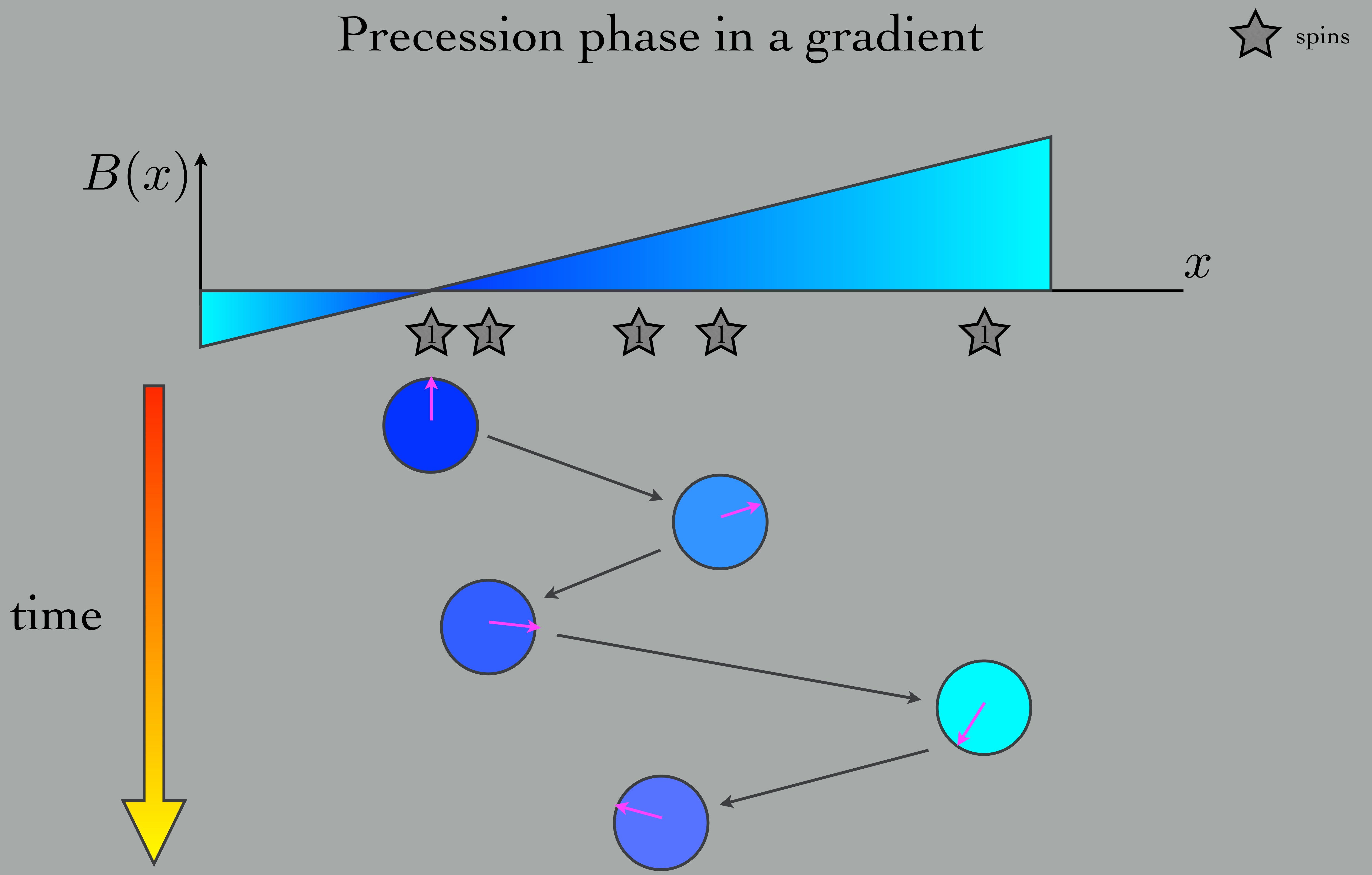
start

end

Three-dimensional

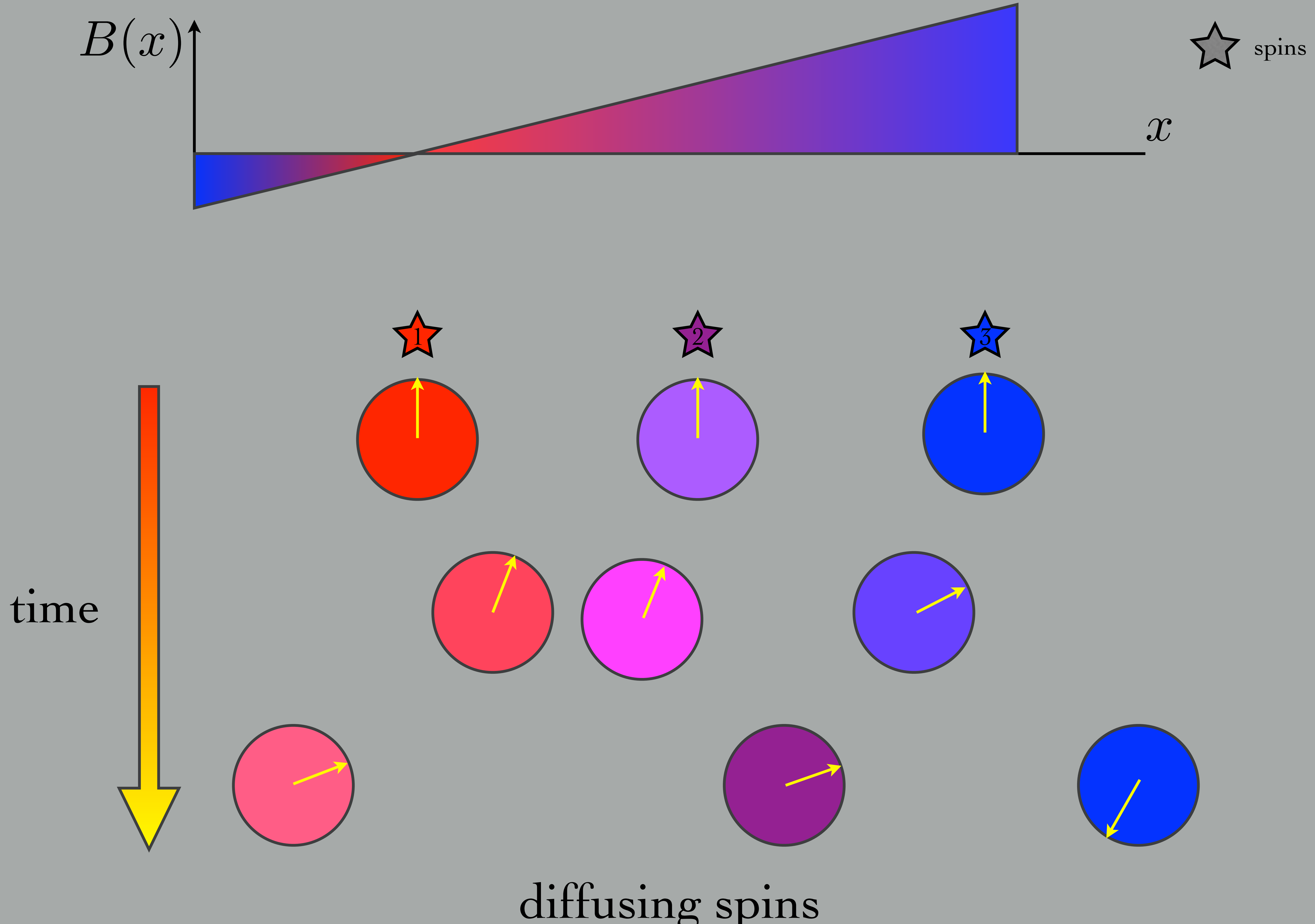
The Basic Ingredient #1:

Precession phase in a gradient

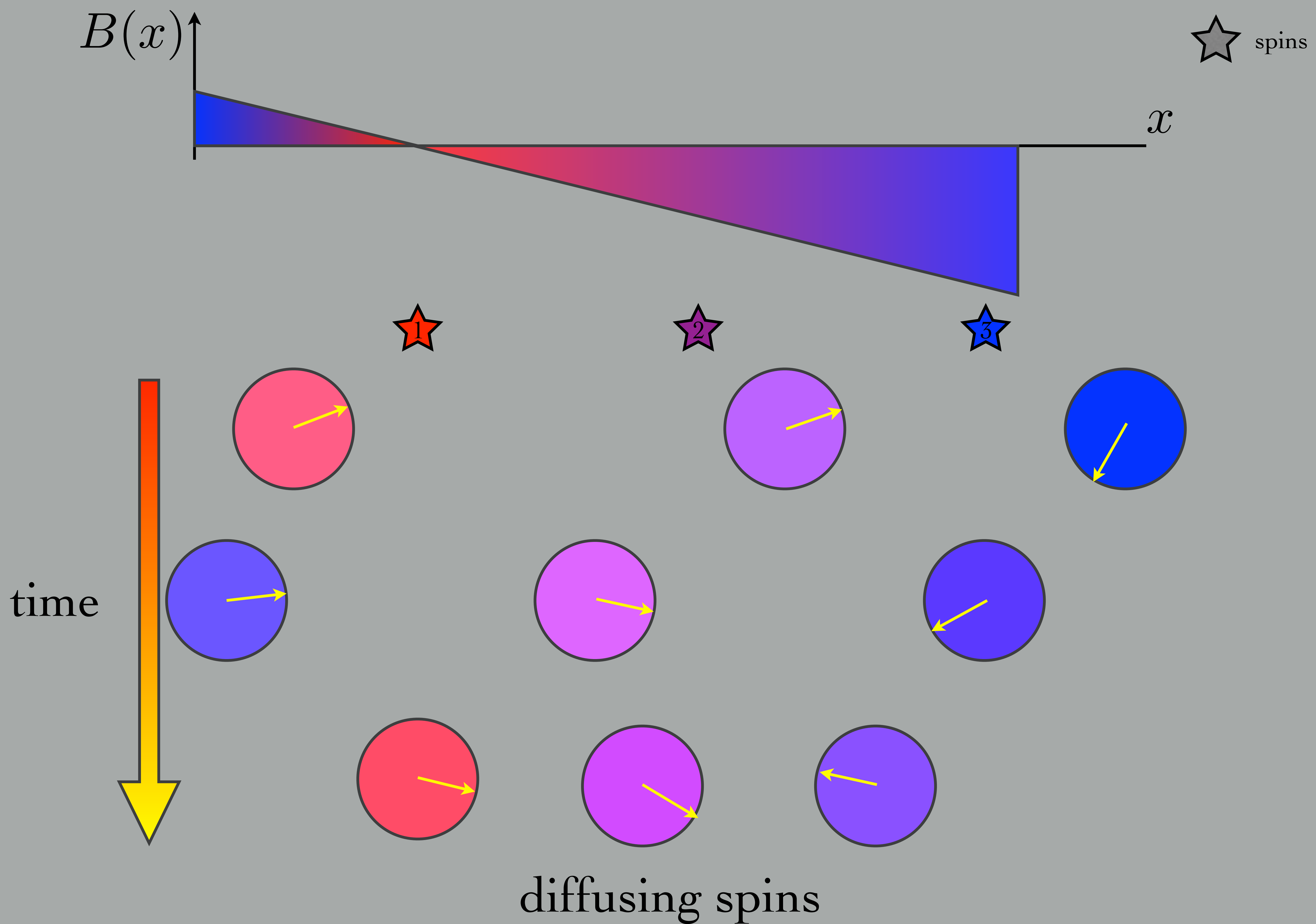


spin doing a random walk

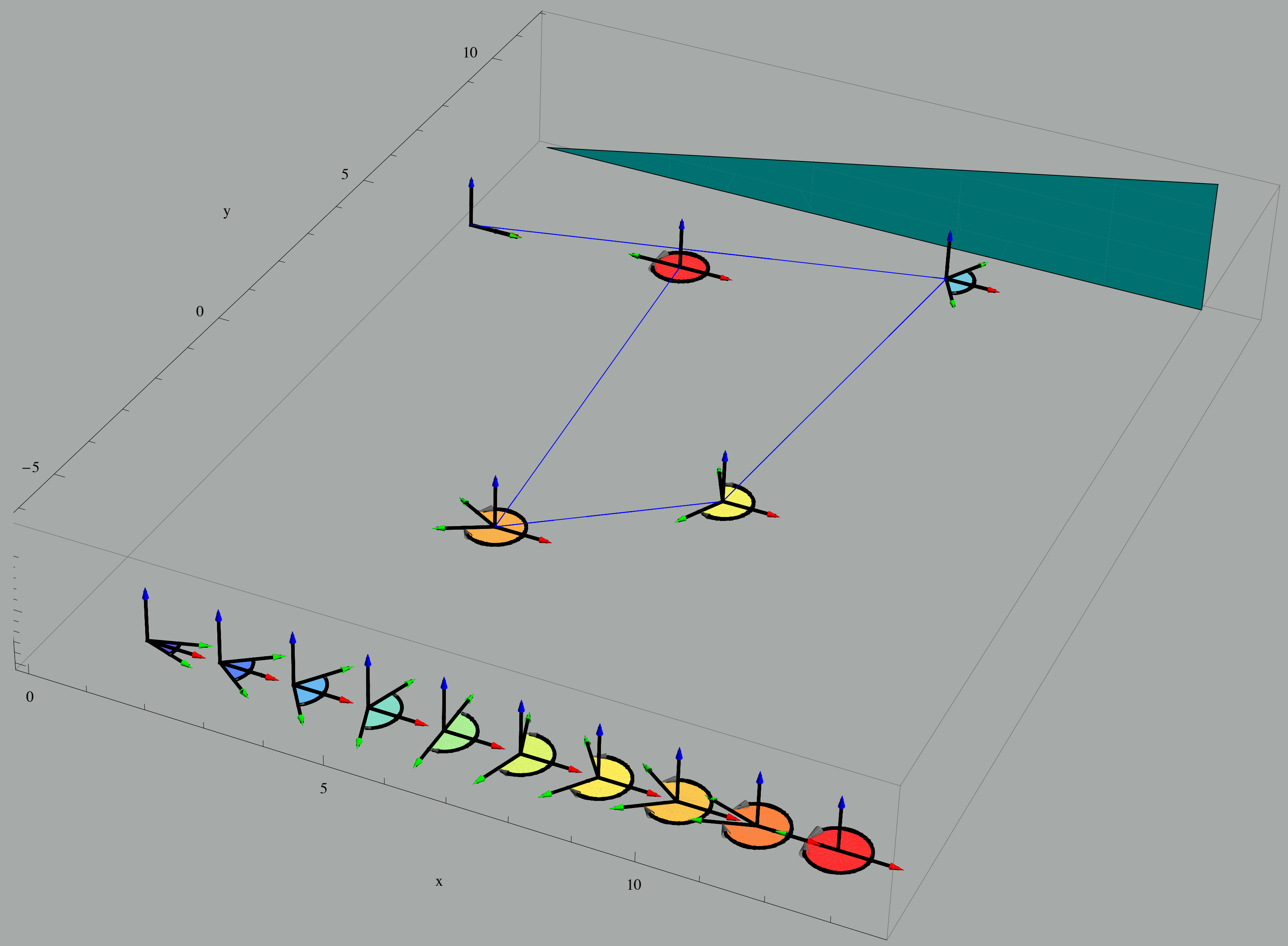
Precession phase in a positive gradient



Precession phase in a positive gradient



The Random Walk in a Gradient



The phase of diffusing spins

$$\varphi(\tau) = \int_0^\tau G(t)x(t)\,dt$$

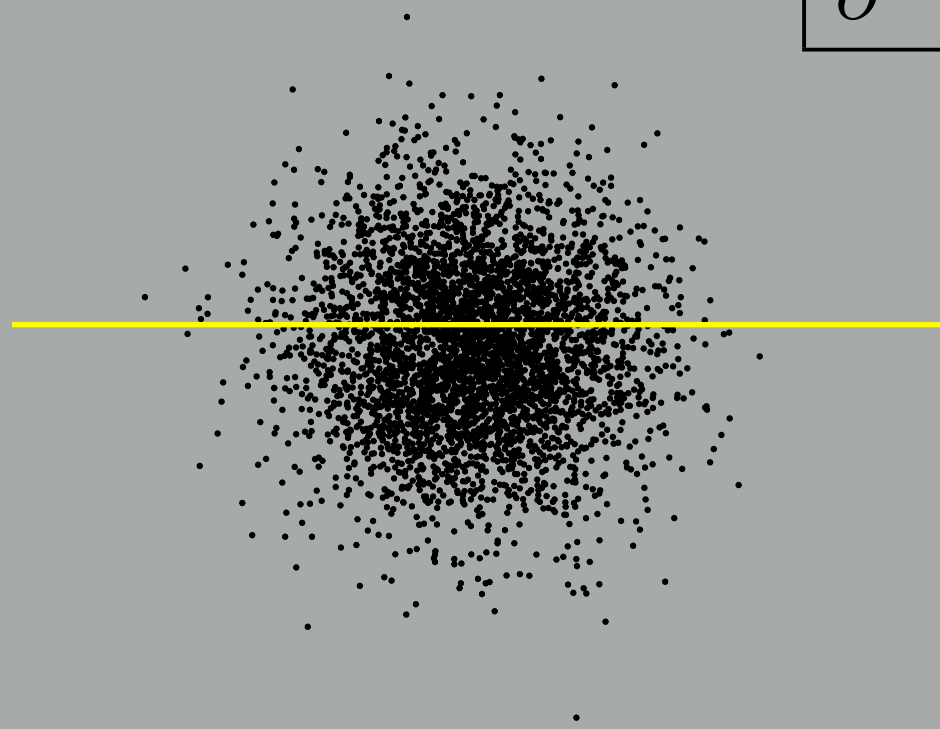
but now motion of x is random

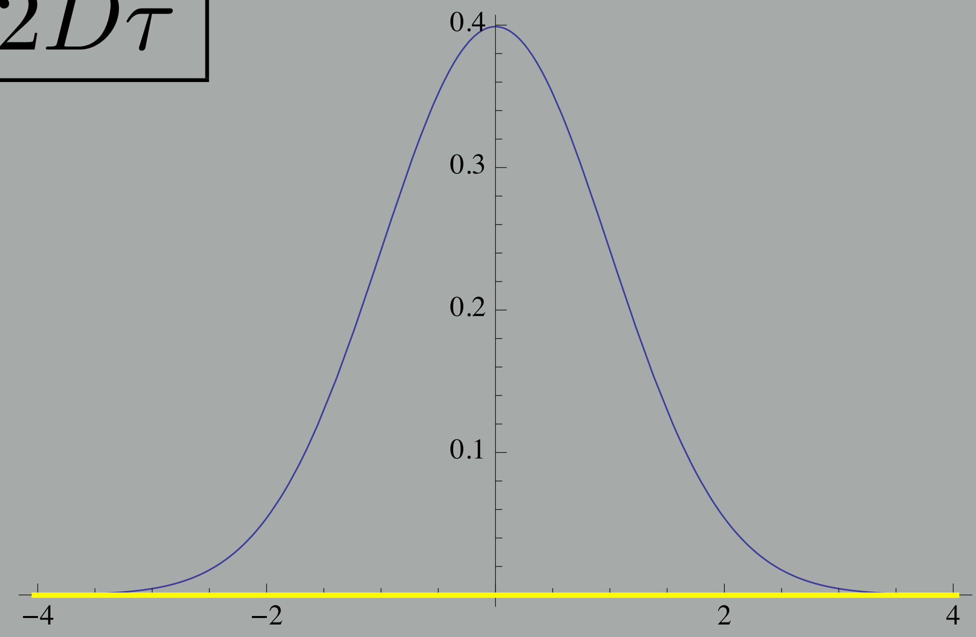
$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Random walk in 1D

$$P(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]$$

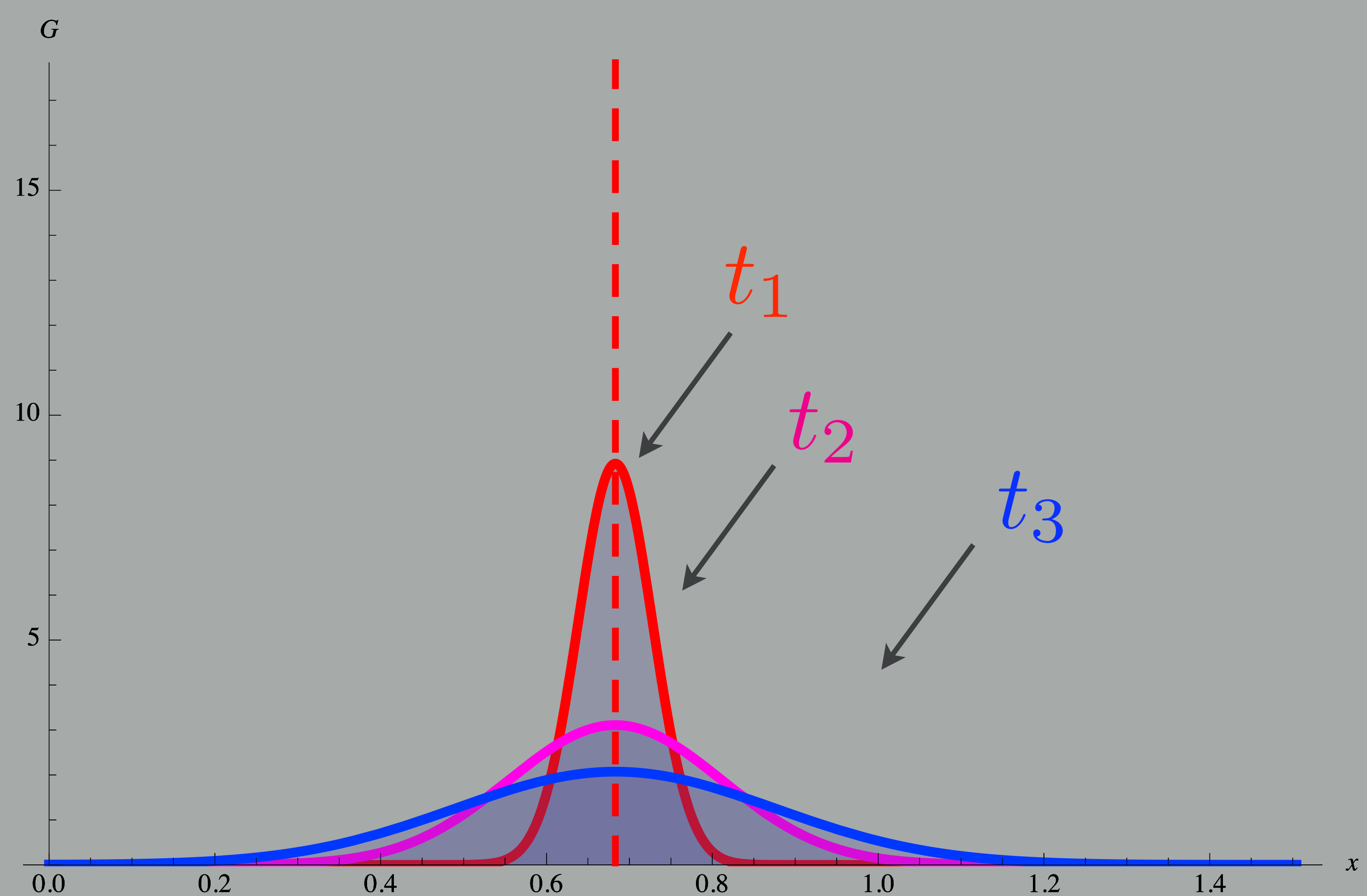
$$\boxed{\sigma^2 = 2D\tau}$$



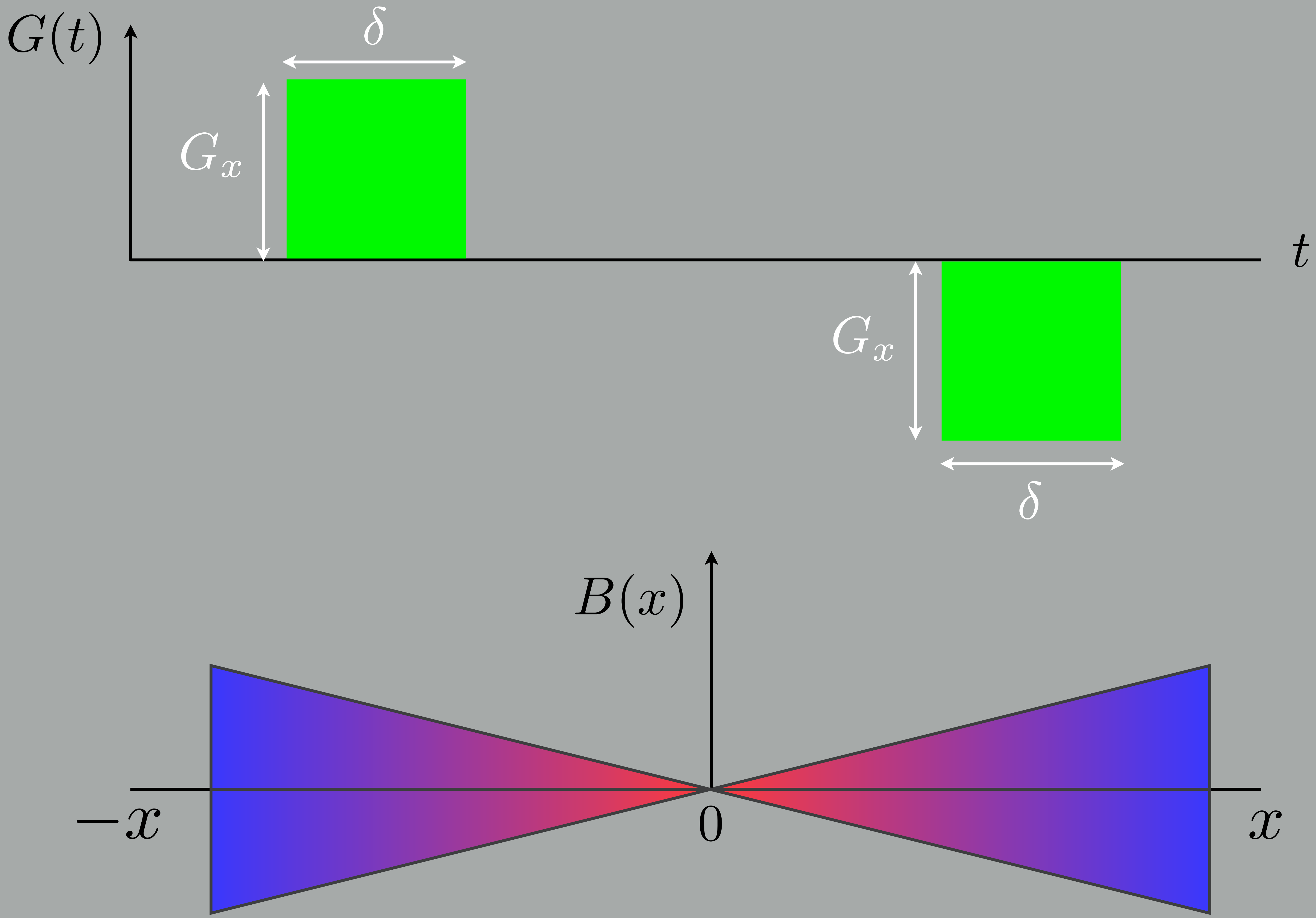


Distribution of particles

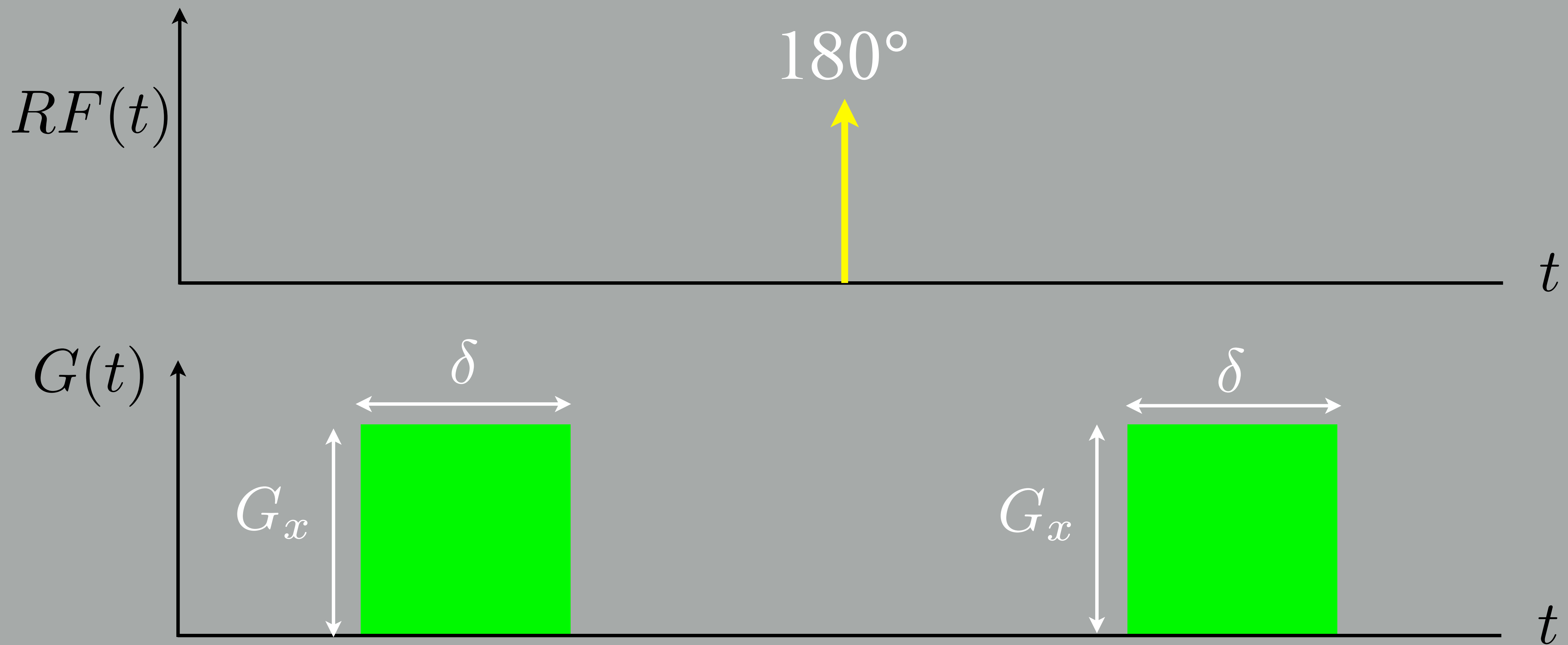
$$P(x) = \frac{1}{(4\pi D\tau)^{1/2}} \exp\left[-\frac{(x-\mu)^2}{4D\tau}\right]$$



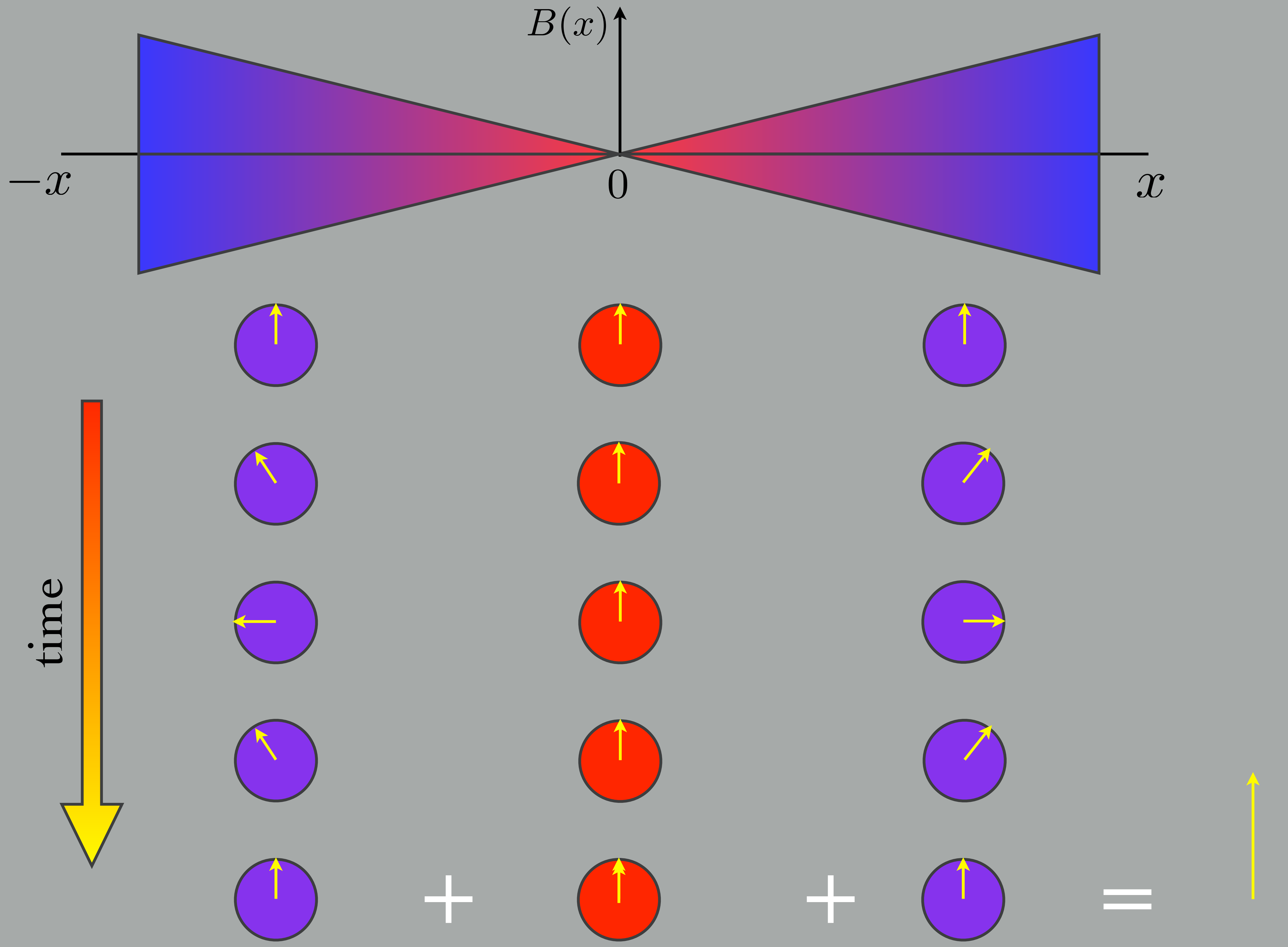
The Bipolar Gradient Pulse (gradient echo)



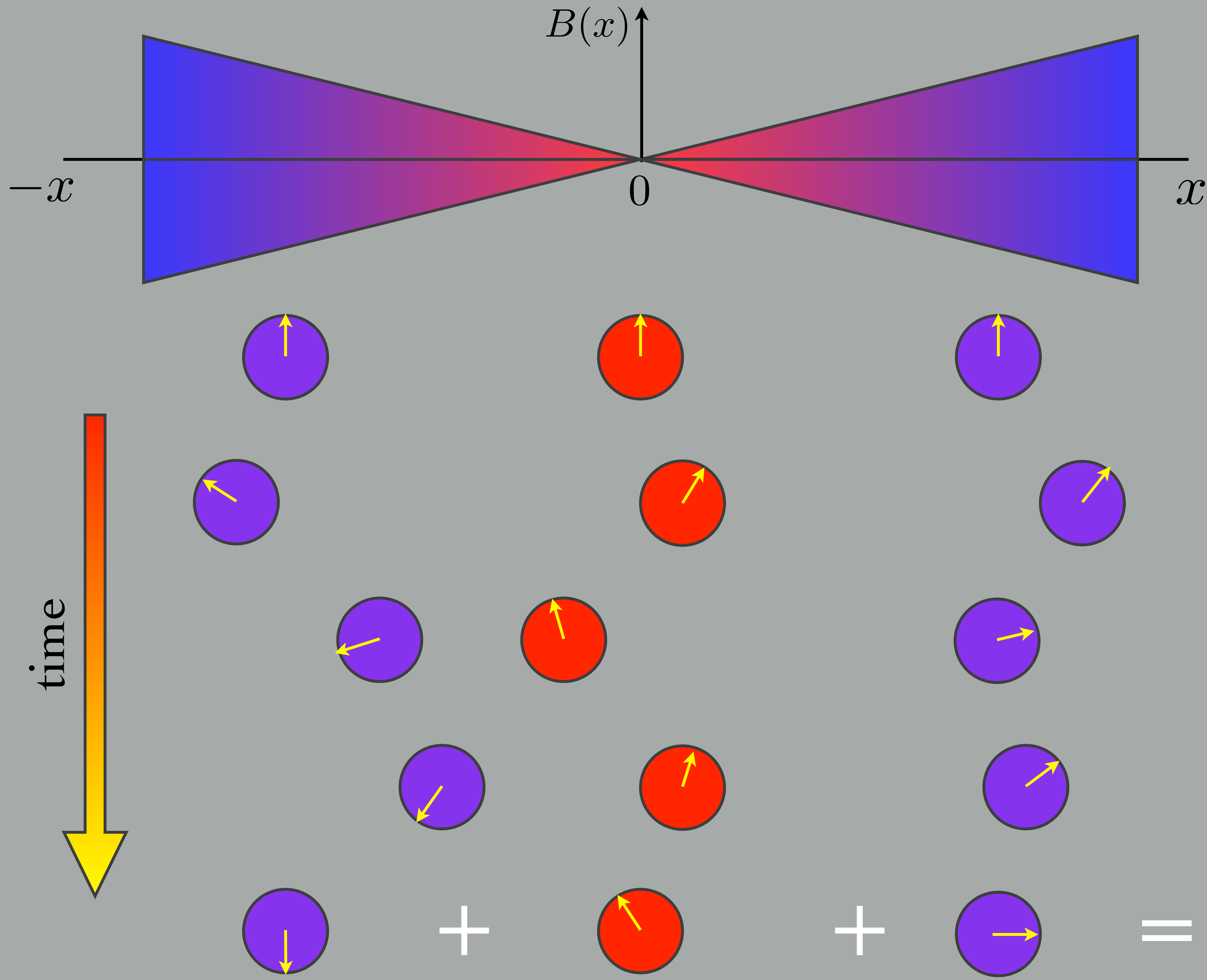
The Bipolar Gradient Pulse (spin echo)



Stationary Spins in Bipolar Pulse



Diffusing Spins in Bipolar Pulse



Key Fact

Only diffusion along the direction of the applied gradient has an effect

Diffusion process is *irreversible*

The complex trajectory followed on one side of the pulse will not be the same on the other side

early NMR measurements of Diffusion

PHYSICAL REVIEW VOLUME 80, NUMBER 4 NOVEMBER 15, 1950

Spin Echoes*†

E. L. Hahn‡
Physics Department, University of Illinois, Urbana, Illinois
(Received May 22, 1950)

Intense radiofrequency power in the form of pulses is applied to an ensemble of spins in a liquid placed in a large static magnetic field H_0. The frequency of the pulsed r-f power satisfies the condition for nuclear magnetic resonance, and the pulses last for times which are short compared with the time in which the nutating macroscopic magnetic moment of the entire spin ensemble can decay. After removal of the pulses a non-equilibrium configuration of isochromatic macroscopic moments remains in which the moment vectors precess freely. Each moment vector has a magnitude at a given precession frequency which is determined by the distribution of Larmor frequencies imposed upon the ensemble by inhomogeneities in H_0. At times determined by pulse sequences applied in the past the constructive interference of these moment vectors gives rise to observable spontaneous nuclear induction signals. The properties and underlying principles of these spin echo signals are discussed with use of the Bloch theory. Relaxation times are measured directly and accurately from the measurement of echo amplitudes. An analysis includes the effect on relaxation measurements of the self-diffusion of liquid molecules which contain resonant nuclei. Preliminary studies are made of several effects associated with spin echoes, including the observed shifts in magnetic resonance frequency of spins due to magnetic shielding of nuclei contained in molecules.

Since there is an established gradient of the magnetic field over the volume of the sample, a molecule whose nuclear moment has been flipped initially in a field H_0, may, in the course of time 2τ, drift by Brownian motion into a randomly differing field H_0. Therefore, as τ is increased, a lesser number of moments participate in the generation of in-phase nuclear radio-frequency signals.

Early Phase Contrast NMR

Journal of Geophysical Research Volume 65, No. 2 February 1960

Detection of Sea-Water Motion by Nuclear Precession

E. L. Hahn

University of California
*LaJolla, California**

$$V(t) = \exp\left\{j\gamma \int_0^t \left[H(x_0) + \left.\frac{dH}{dx}\right|_{x_0} vt \right] dt\right\}$$

$$= \exp j\gamma[H(x_0)t + Gvt^2/2]$$

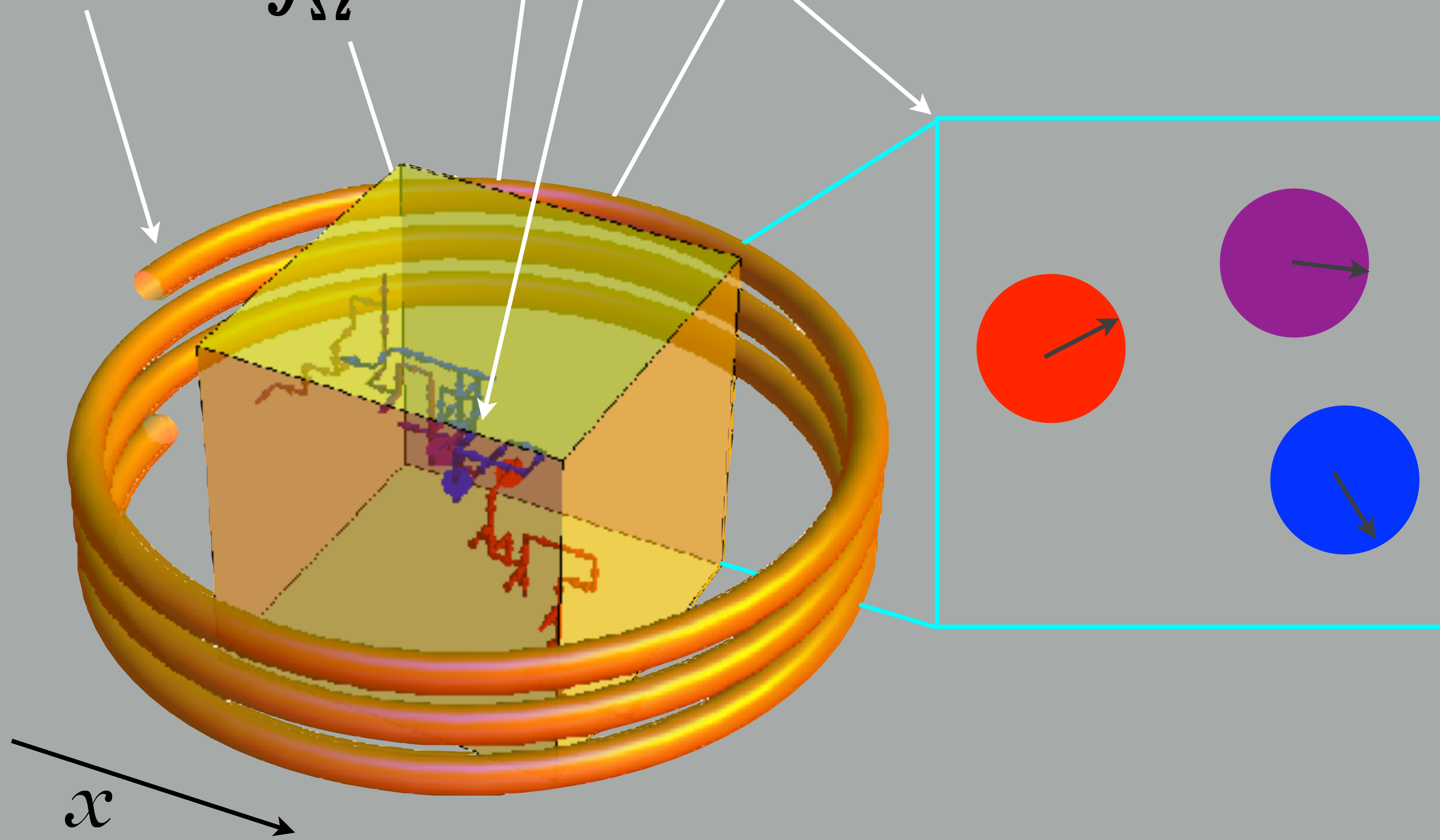
where $G = (dH/dx)_{x_0}$.

We see that the presence of a velocity v produces a phase shift in time t given by $\Delta\varphi = \gamma G v t^2/2$ (assuming a constant field gradient).

signal = sum over all spins

$$S(\varphi) = \int_{\Omega} dx\, \rho(x)\, e^{-i\varphi(x,t)}$$

spins (mass, location, phase)



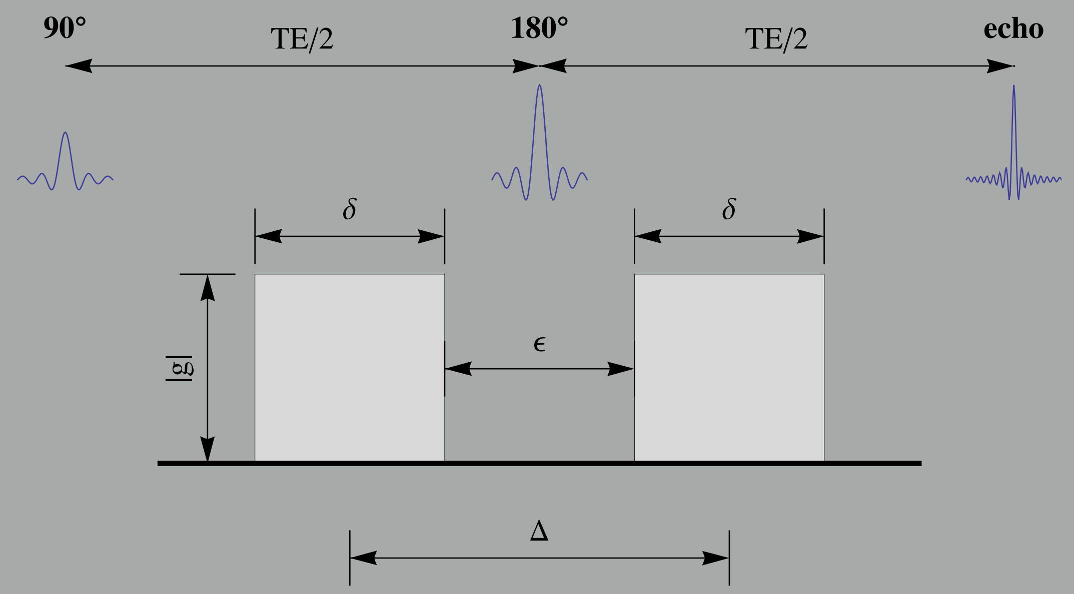
The NMR signal

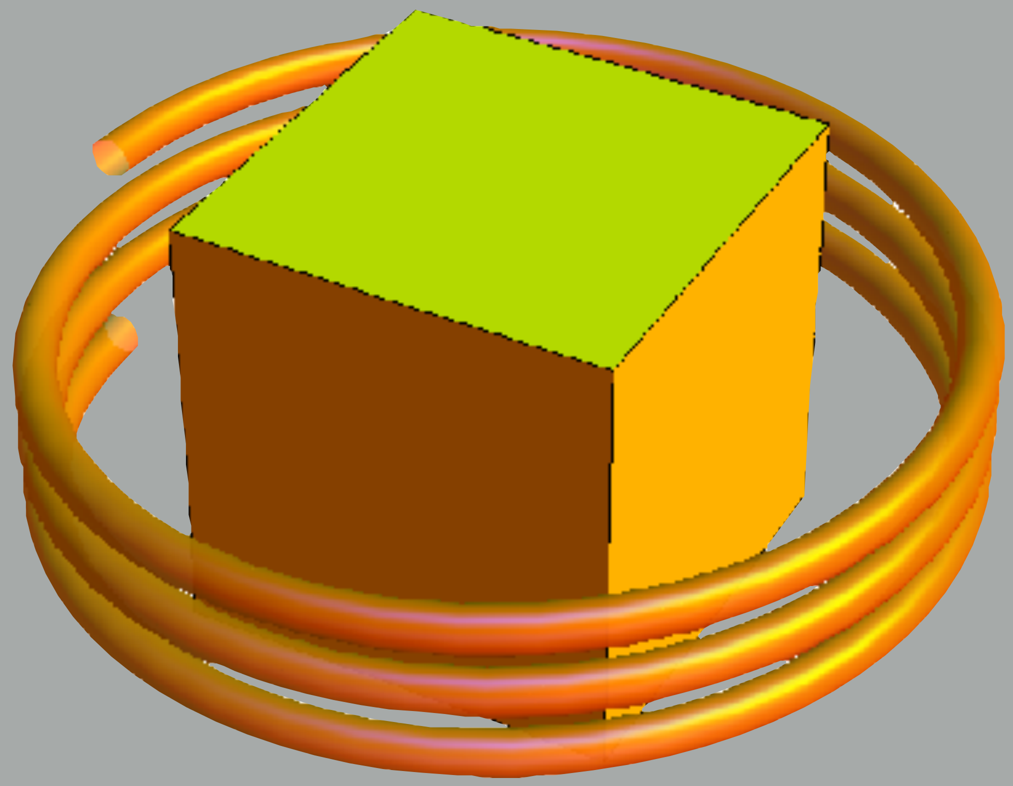
$$s(\varphi) = \int_\Omega m_\perp(\boldsymbol{x},t)\, e^{-i\varphi(\boldsymbol{x},t)}\, d\boldsymbol{x}$$

$$m_\perp(\boldsymbol{x},t) \to P(\boldsymbol{x},t)$$

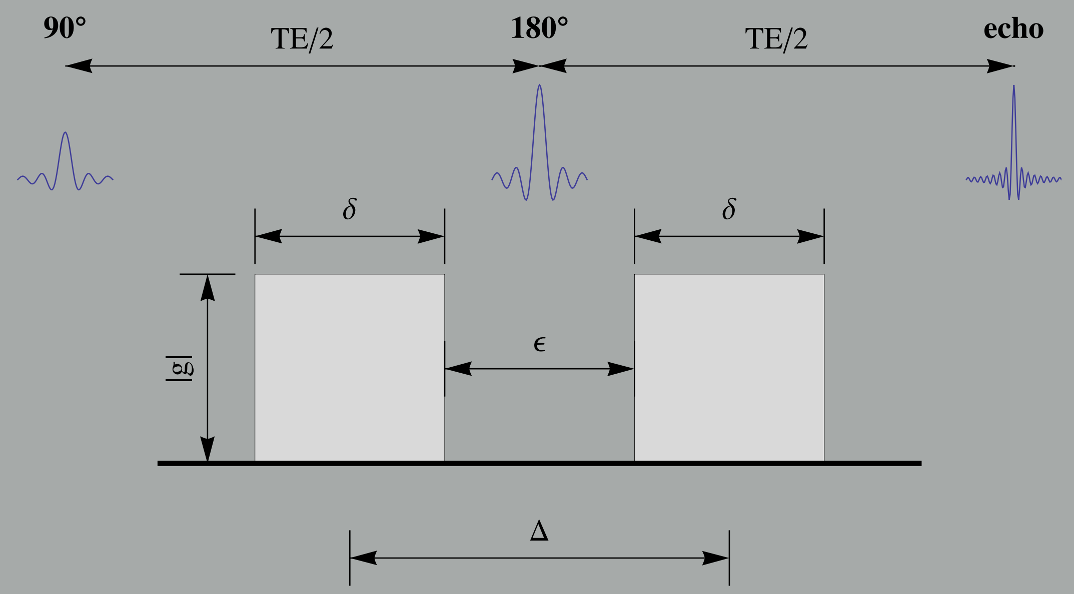
$$s(\varphi) = \int_\Omega P(\boldsymbol{x},t)\, e^{-i\varphi(\boldsymbol{x},t)}\, d\boldsymbol{x}$$

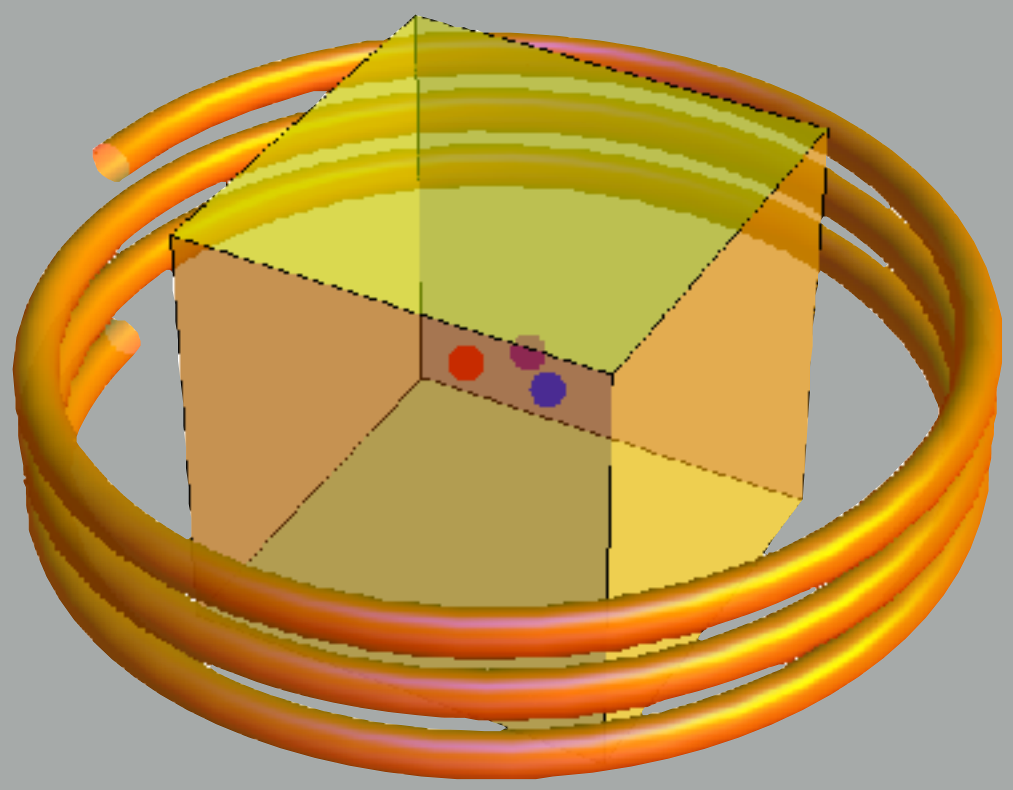
Basic Diffusion Coefficient Experiment



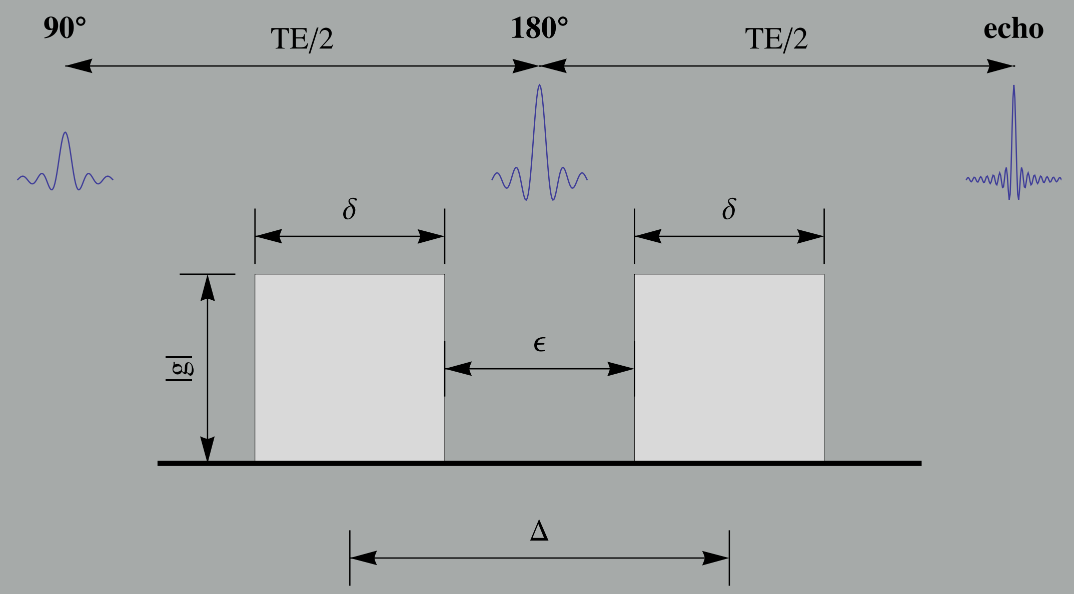


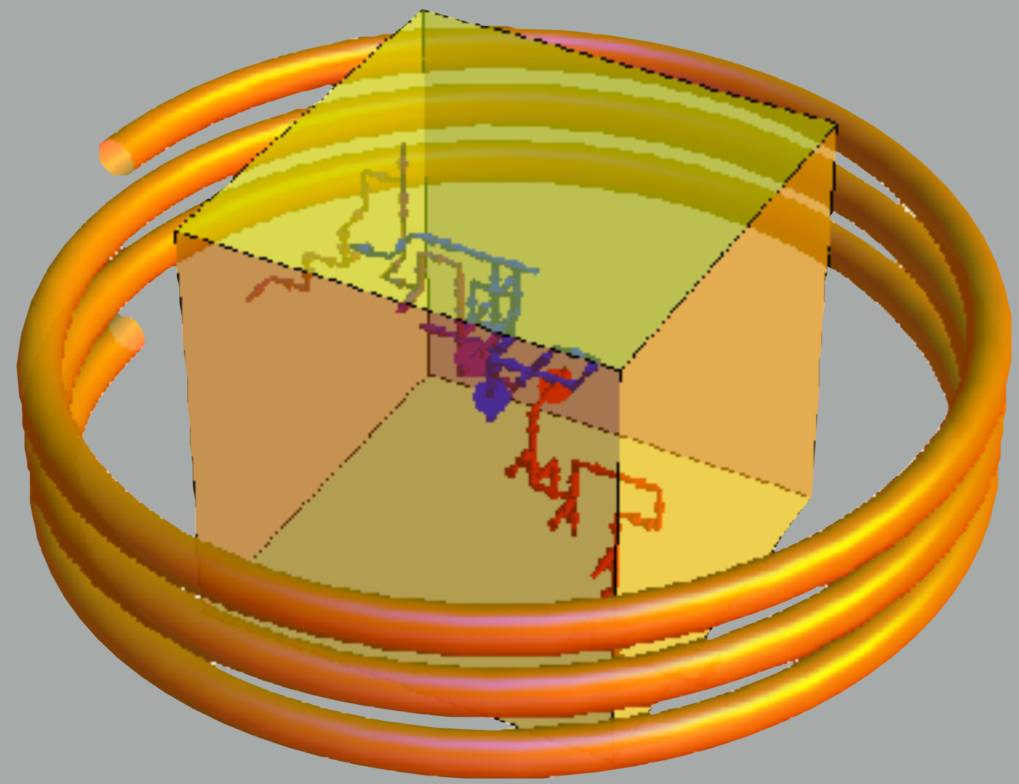
Basic Diffusion Coefficient Experiment

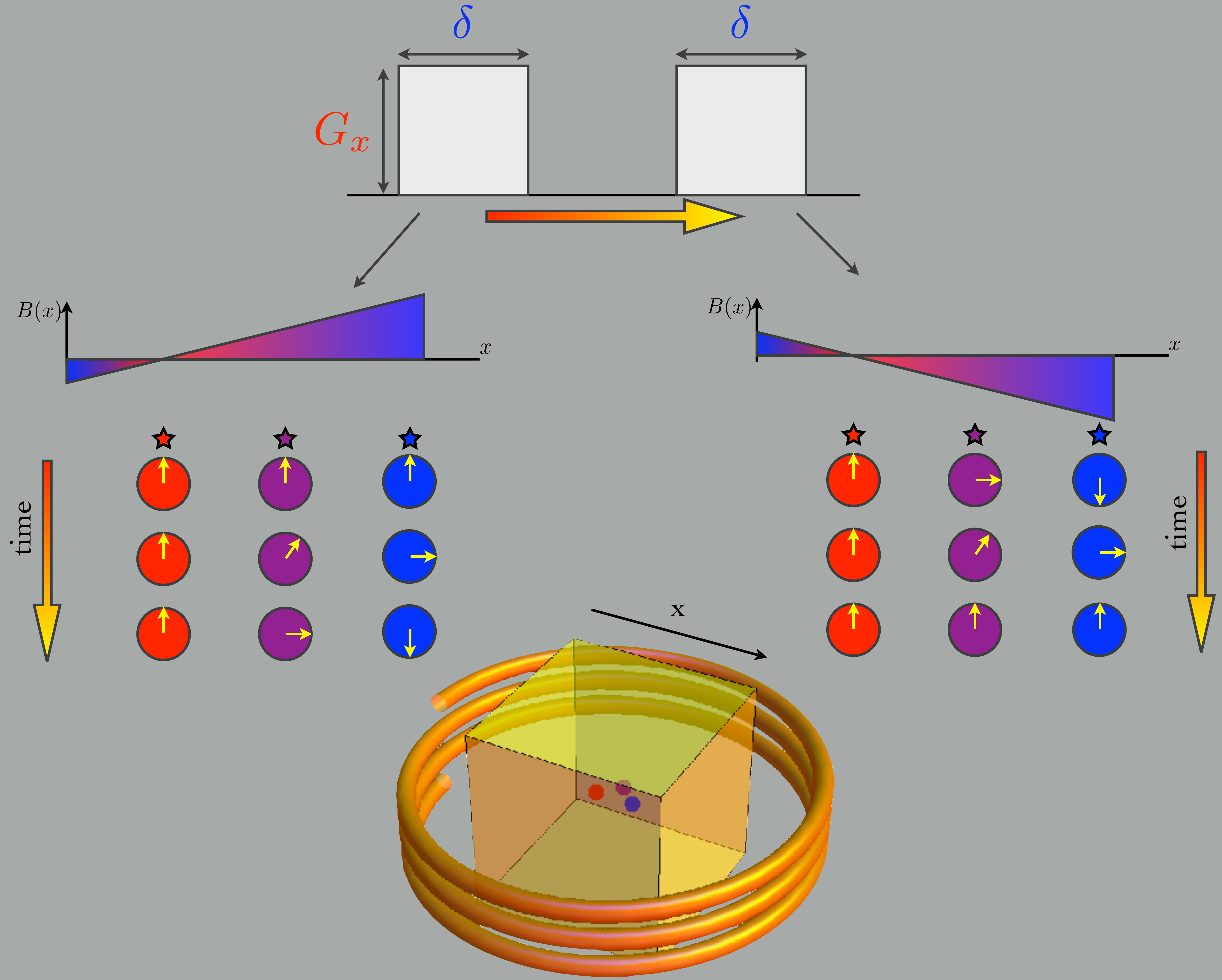




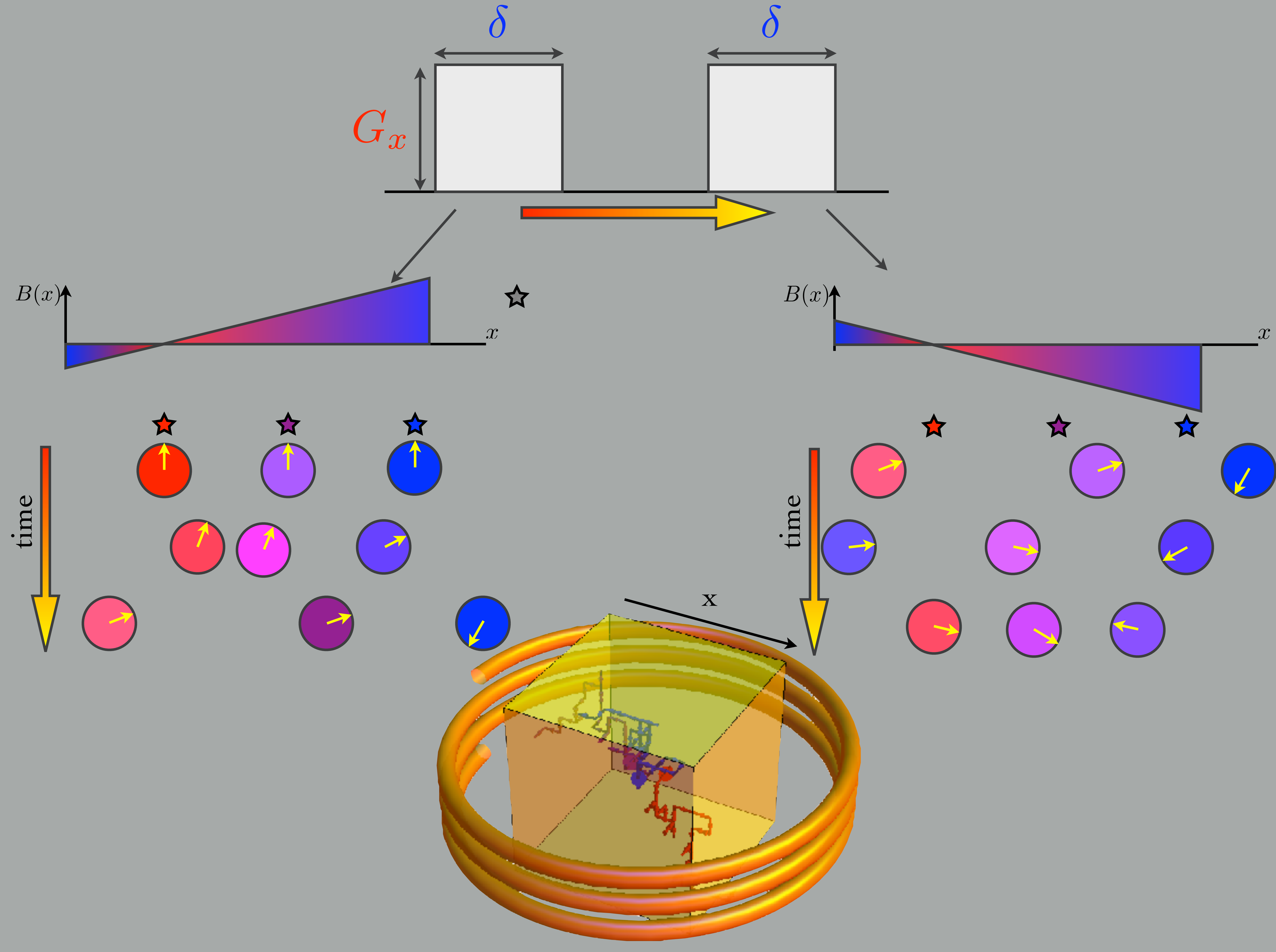
Basic Diffusion Coefficient Experiment





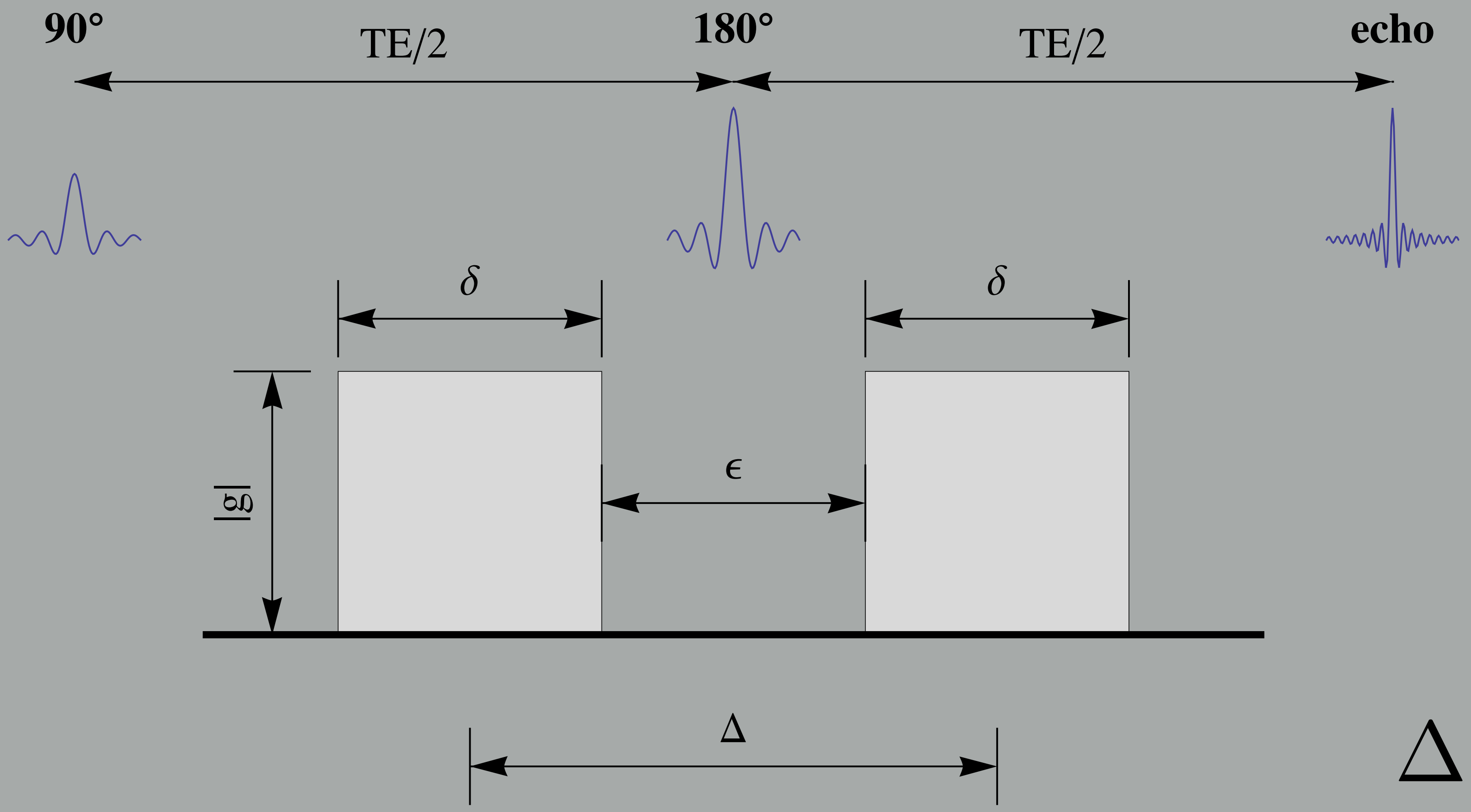


δ
δ
G_x
$B(x)$
x
$B(x)$
x
time
time
x



δ
δ
G_x
$B(x)$
x
$B(x)$
x
time
time
x

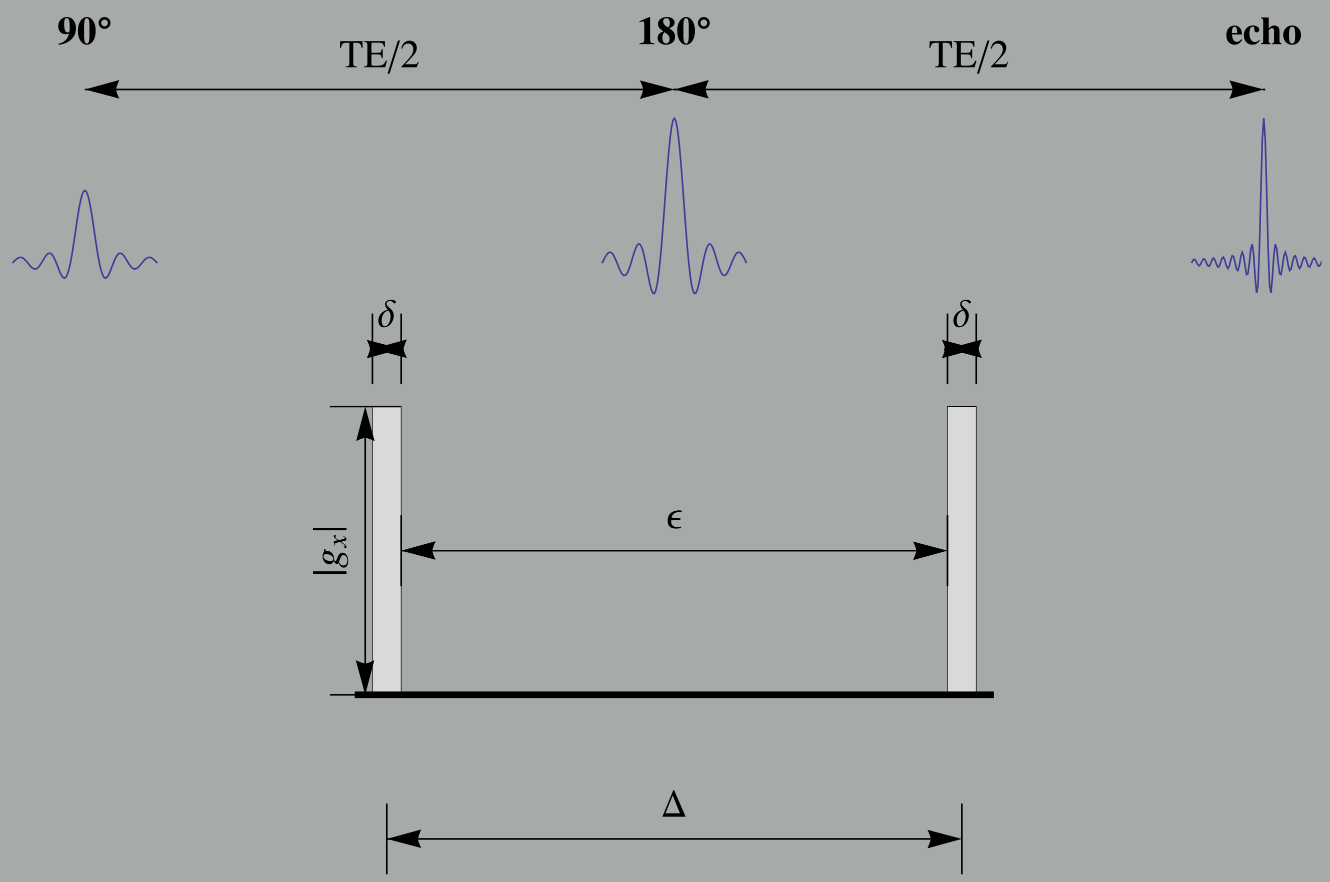
Phases of diffusing spins



$$\Delta = \delta + \epsilon$$

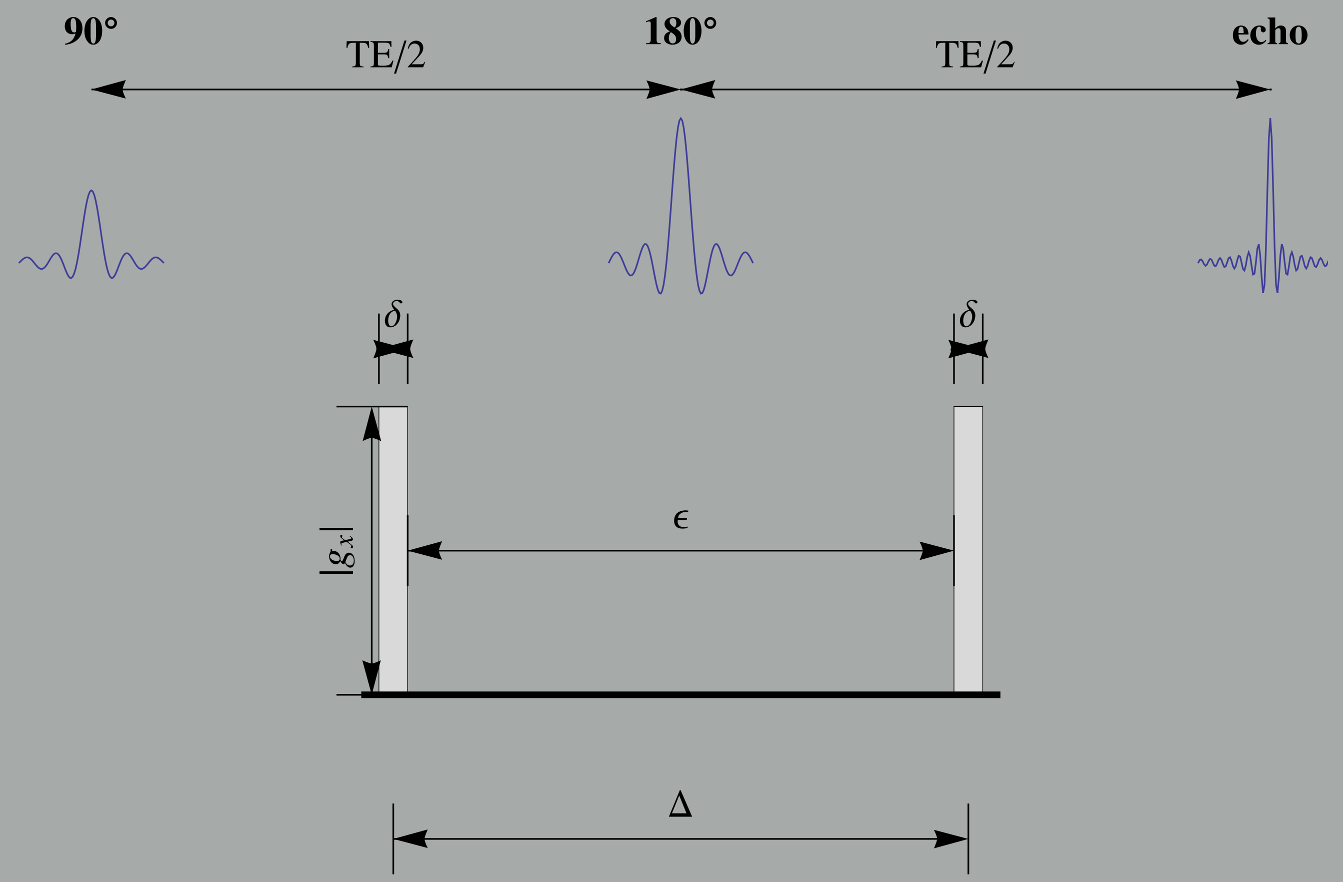
$$\varphi(\tau) = \int_0^\tau G(t)x(t)\,dt$$

The narrow pulse



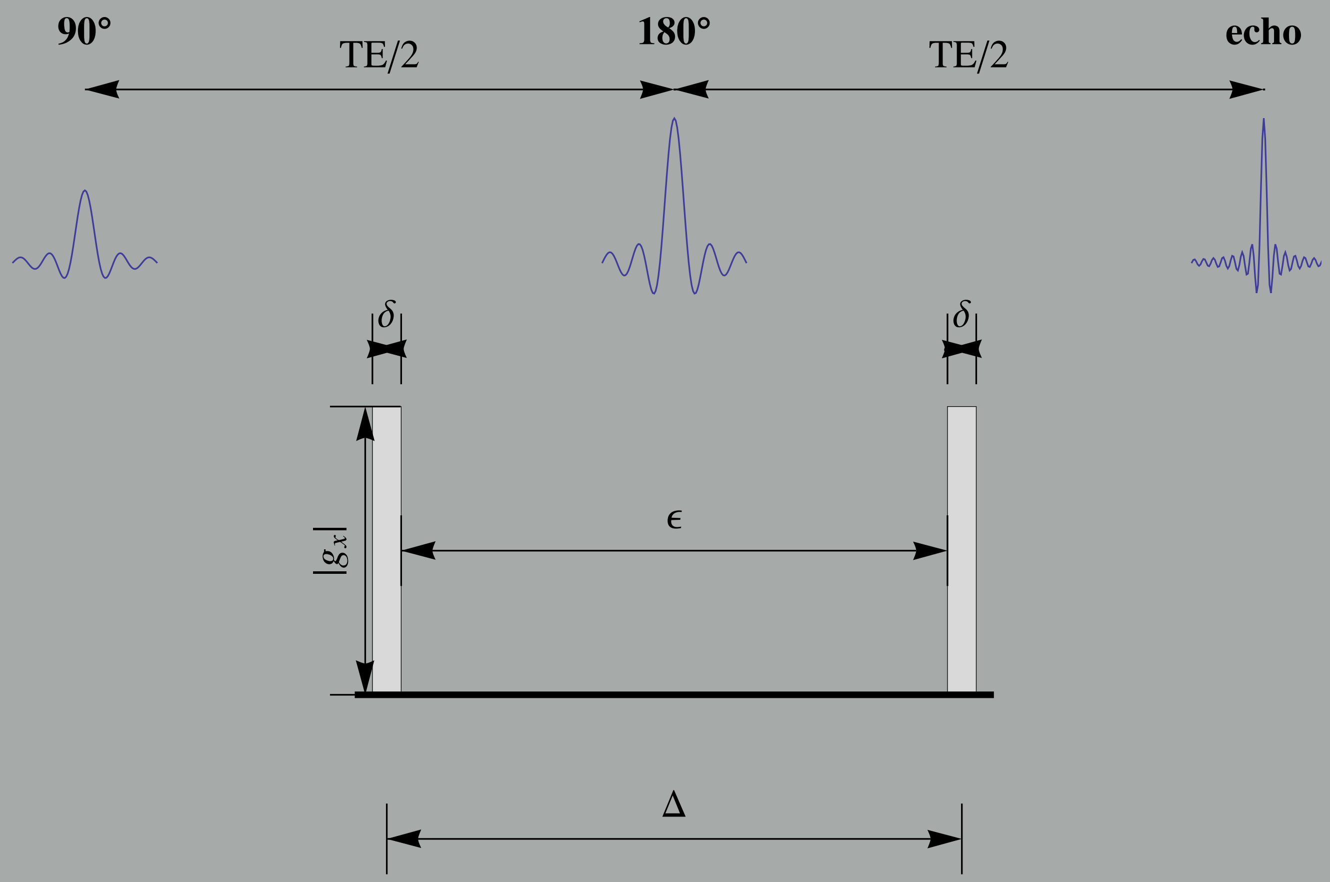
$$\varphi(\tau) = \int_0^\tau G(t)x(t)\,dt$$

The narrow pulse



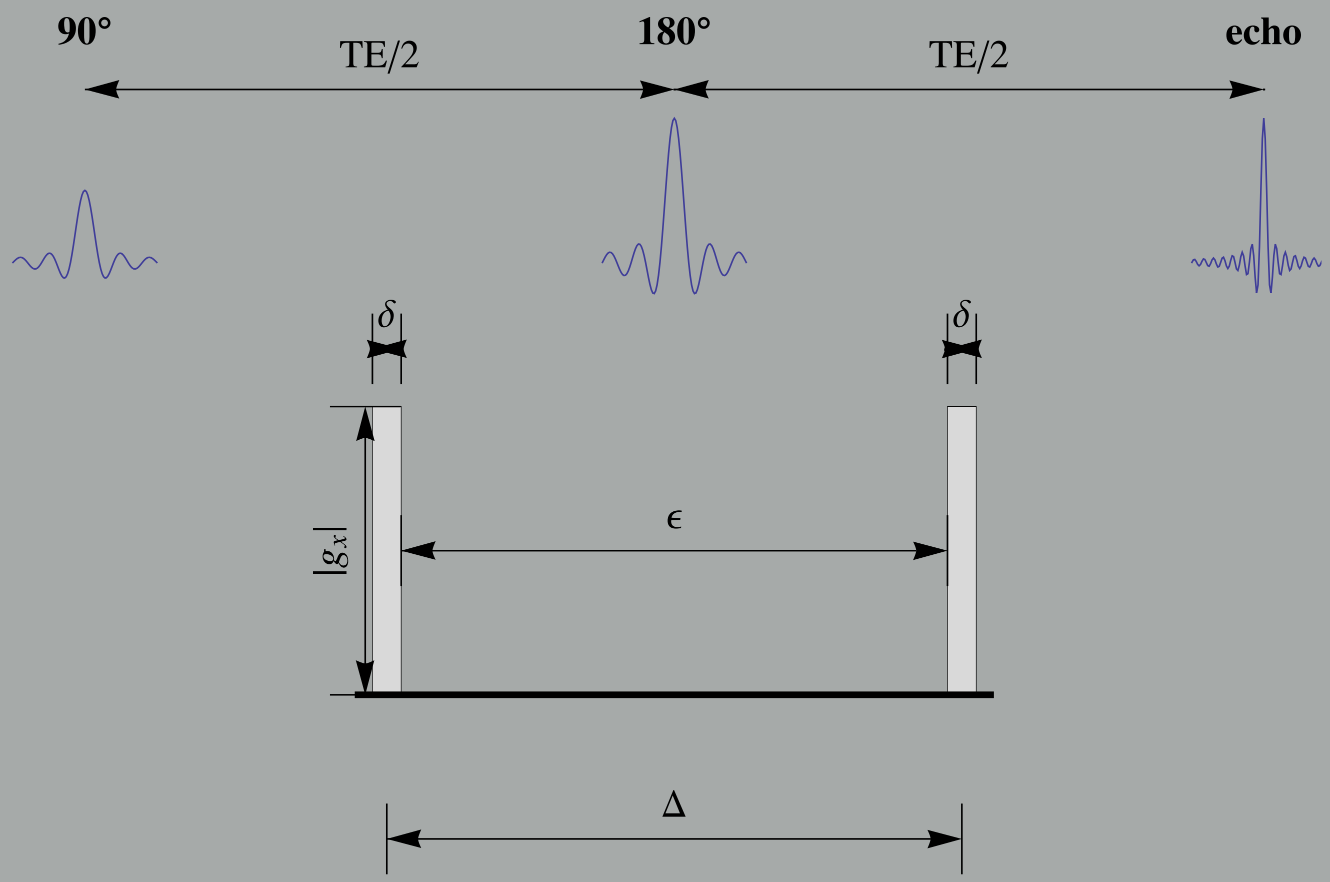
$$\varphi(\tau) = -G\delta\ x(t_1) + G\delta\ x(t_2)$$

The narrow pulse approximation



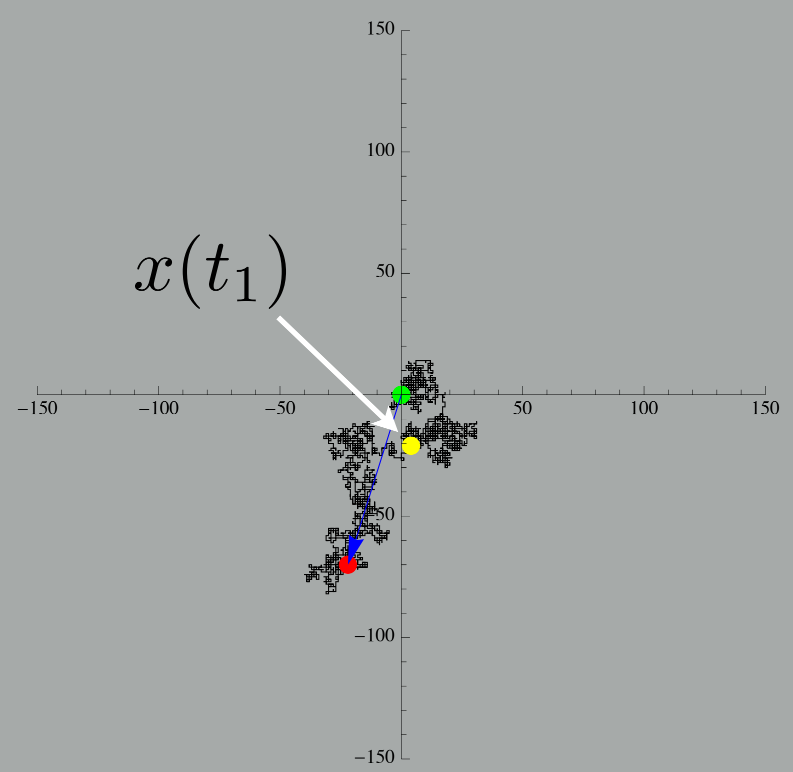
$$\varphi(\tau) = q\underbrace{\left[x(t_2) - x(t_1)\right]}_{r} \qquad q = G\delta$$

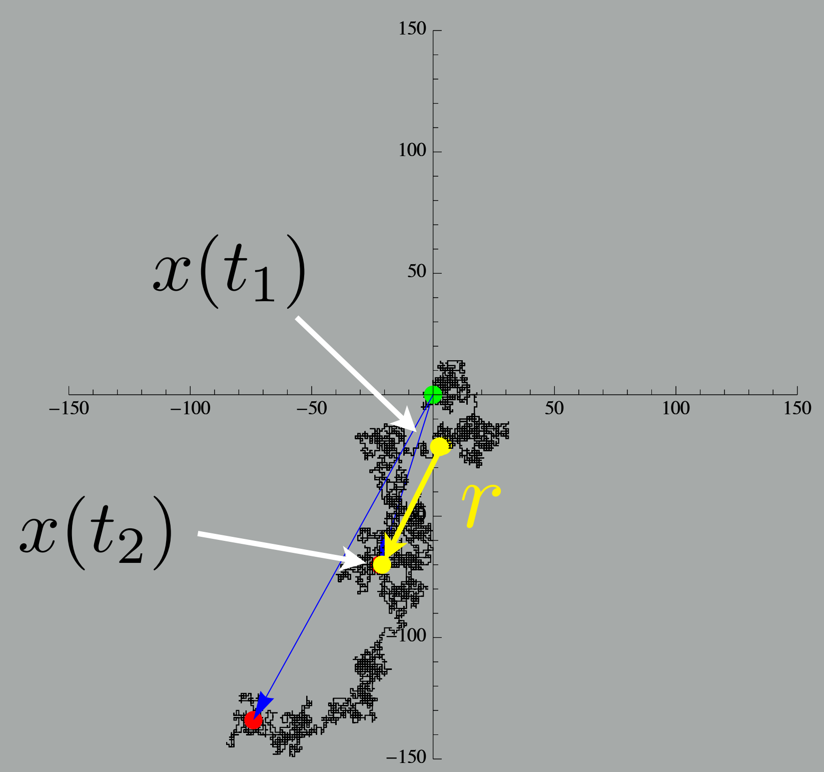
The narrow pulse approximation



$$\varphi(\tau) = qr \text{ where } \begin{cases} q = G\delta \\ r = x(t_2) - x(t_1) \end{cases}$$

The narrow pulse approximation





$$\varphi(\tau) = \underbrace{G\delta}_{q} \underbrace{\left[x(t_2) - x(t_1)\right]}_{r} = qr$$

The NMR signal for the narrow pulse

$$s(\varphi) = \int_{\Omega} P(\boldsymbol{x}, t)\, e^{-i\varphi(\boldsymbol{x}, t)}\, d\boldsymbol{x}$$

where

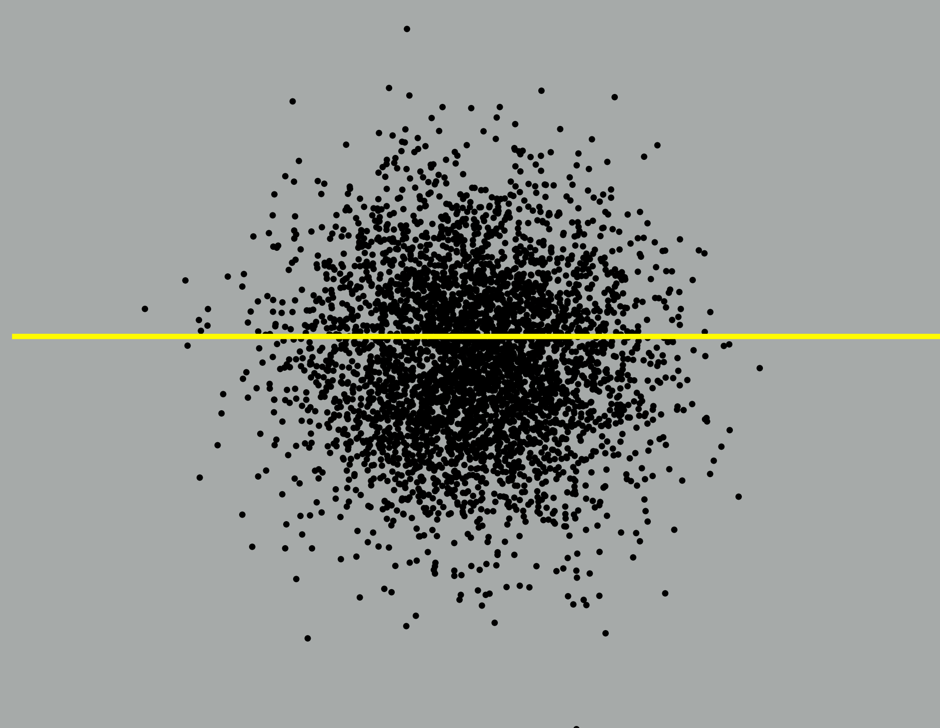
$$\varphi(\tau) = \underbrace{G\delta}_{q}\ \underbrace{[x(t_2) - x(t_1)]}_{r} = qr$$

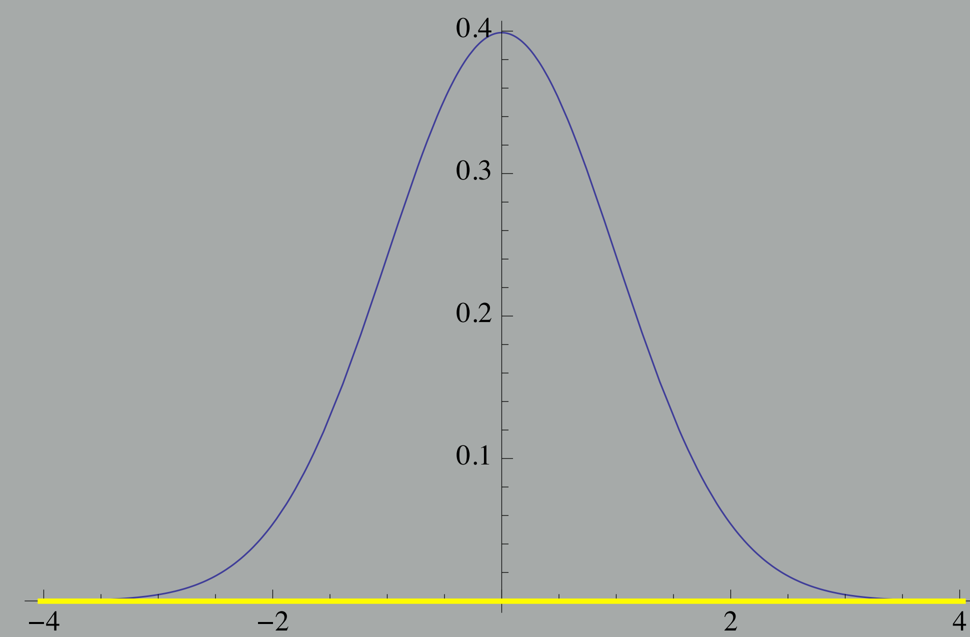
$$s(q, t) = \int P(r, t)\, e^{-iqr}\, dr$$

Recall: For Gaussian diffusion in 1D

$$P(r,t) = \frac{1}{\sqrt{4\pi D\tau}} e^{r^2/(4D\tau)}$$

$$\tau = \Delta - \delta/3$$





The NMR signal for the narrow pulse

$$s(q,t) = \int P(r,t)\, e^{-iqr}\, dr$$

$$P(r,t) = \frac{1}{\sqrt{4\pi D\tau}} e^{r^2/(4D\tau)}$$

$$s(q,t) = \frac{1}{\sqrt{4\pi D\tau}} \int e^{r^2/(4D\tau)}\, e^{-iqr}\, dr$$

The NMR signal for the narrow pulse

$$s(q,t) = \frac{1}{\sqrt{4\pi D\tau}} \int e^{r^2/(4D\tau)}\, e^{-iqr}\, dr$$

$$\boxed{s(q,\tau) = s(0)e^{-bD}}$$

where $\quad b = q^2\tau$

The NMR signal for the narrow pulse

$$s(q,t) = \frac{1}{\sqrt{4\pi D\tau}} \int e^{r^2/(4D\tau)}\, e^{-iqr}\, dr$$

$$\boxed{s(q,\tau) = s(0)e^{-bD}}$$

where $\quad b = q^2\tau = G^2\delta^2(\Delta - \delta/3)$

The NMR signal for the narrow pulse

$$s(q,\tau) = s(0)e^{-bD}$$

where $\quad b = q^2\tau = G^2\delta^2(\Delta - \delta/3)$

The signal is a decaying exponential with amplitude equal to the non-diffusion weighted image s(0) and a decay proportional to the *square* of the diffusion weighting gradient area q times the diffusion time τ (both known) and the diffusion coefficient *D,* which is unknown .

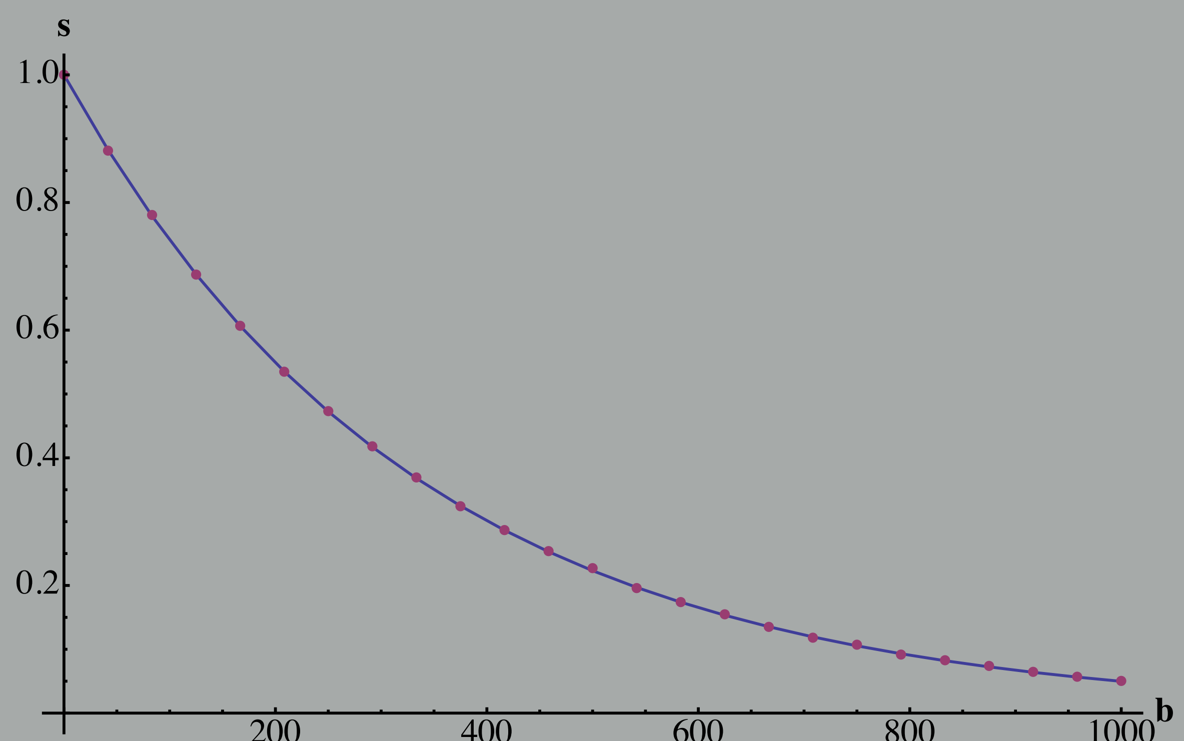
The signal from 1D Gaussian Diffusion

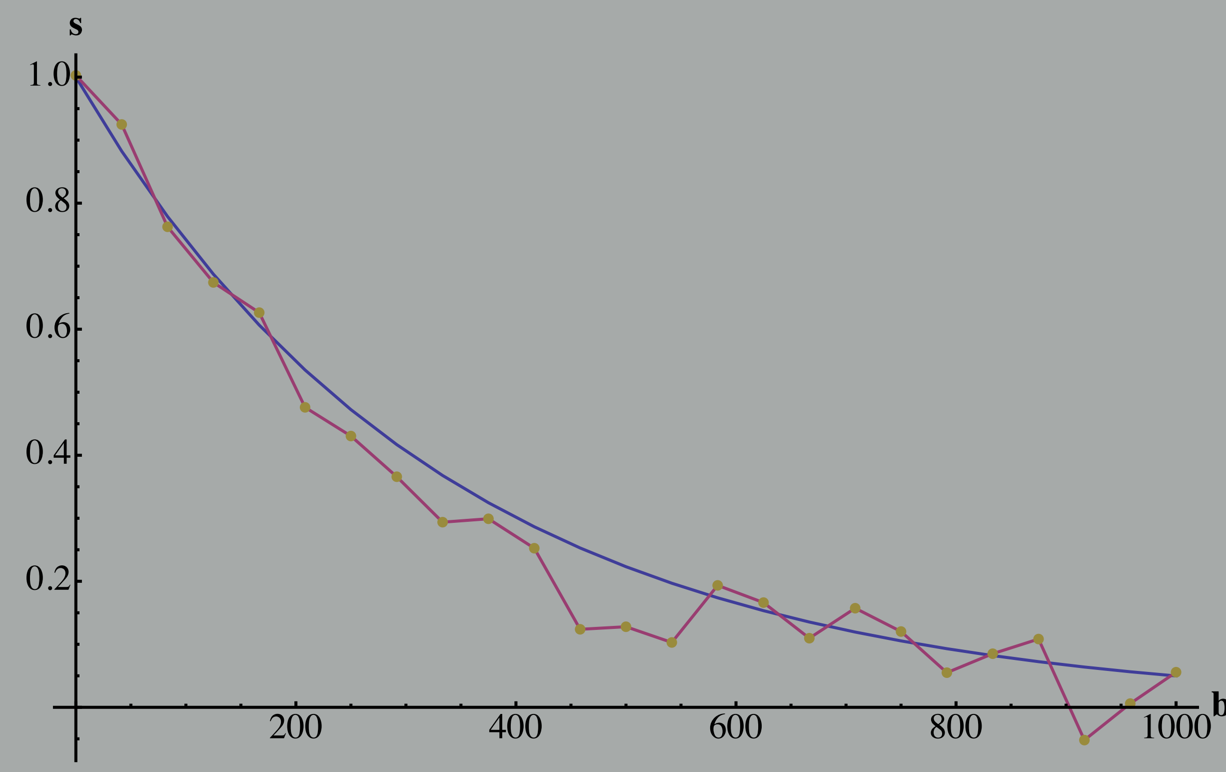
Ideal

$$s(b) = s(0)e^{-bD}$$

Actual

$$s(b) = s(0)e^{-bD} + \eta(b)$$





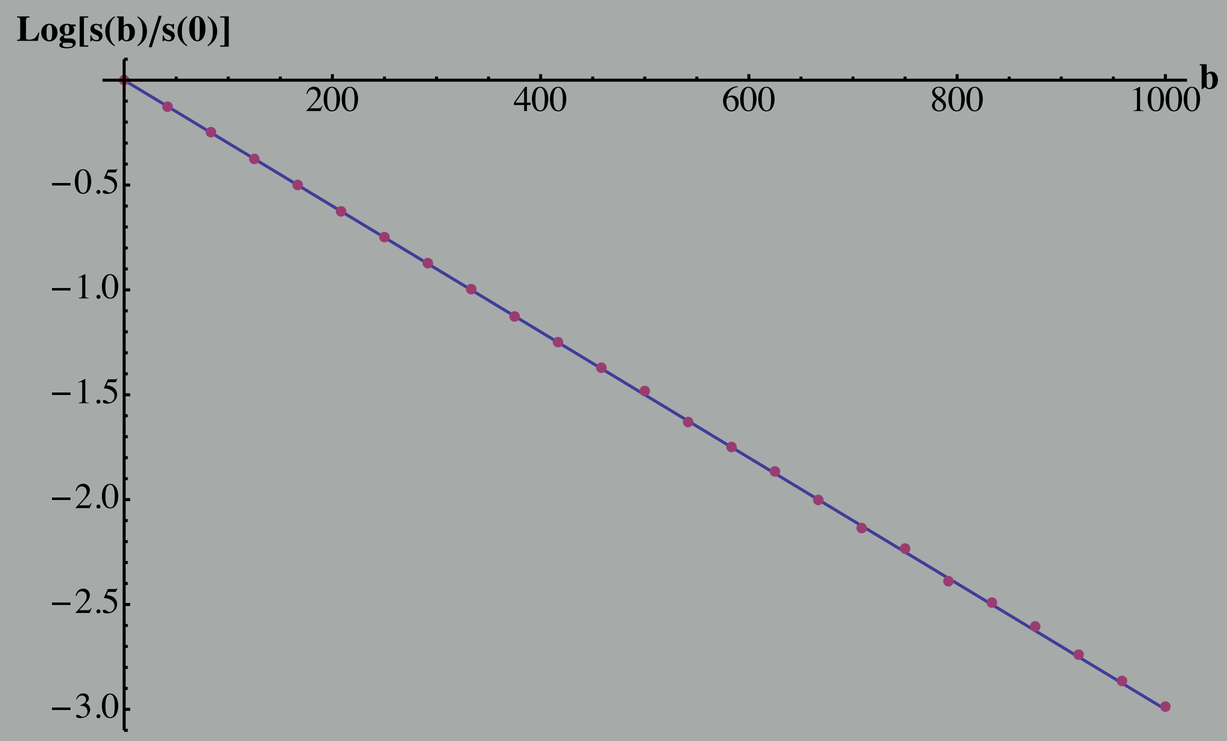
The signal from 1D Gaussian Diffusion

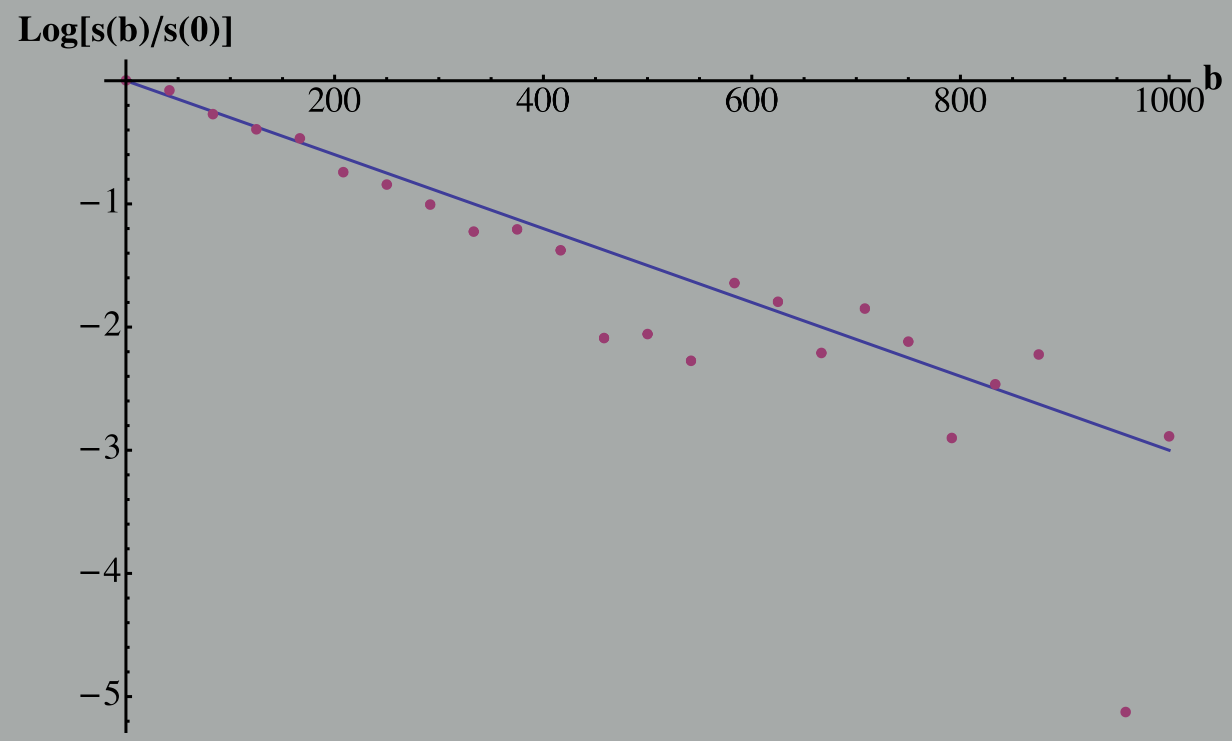
Ideal

$$\log\left[\frac{s(b)}{s(0)}\right] = -bD$$

Actual

$$\log\left[\frac{s(b)}{s(0)}\right] \approx -bD$$

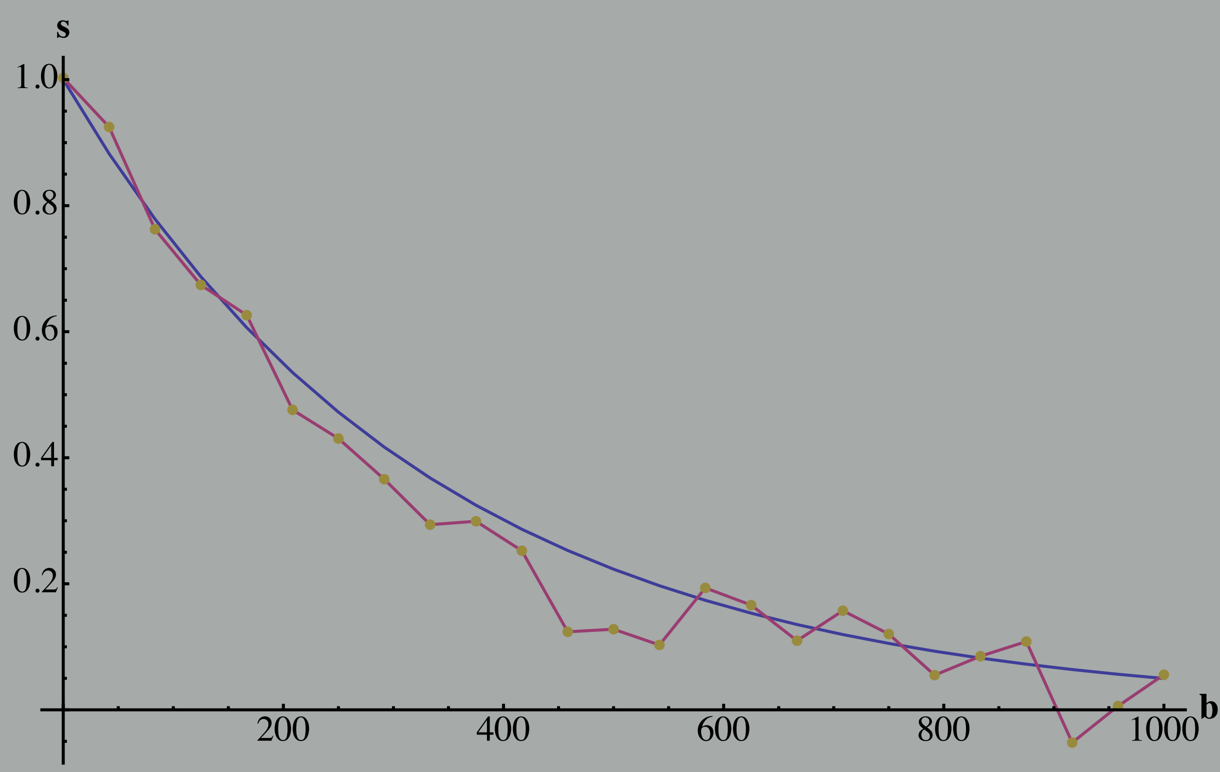


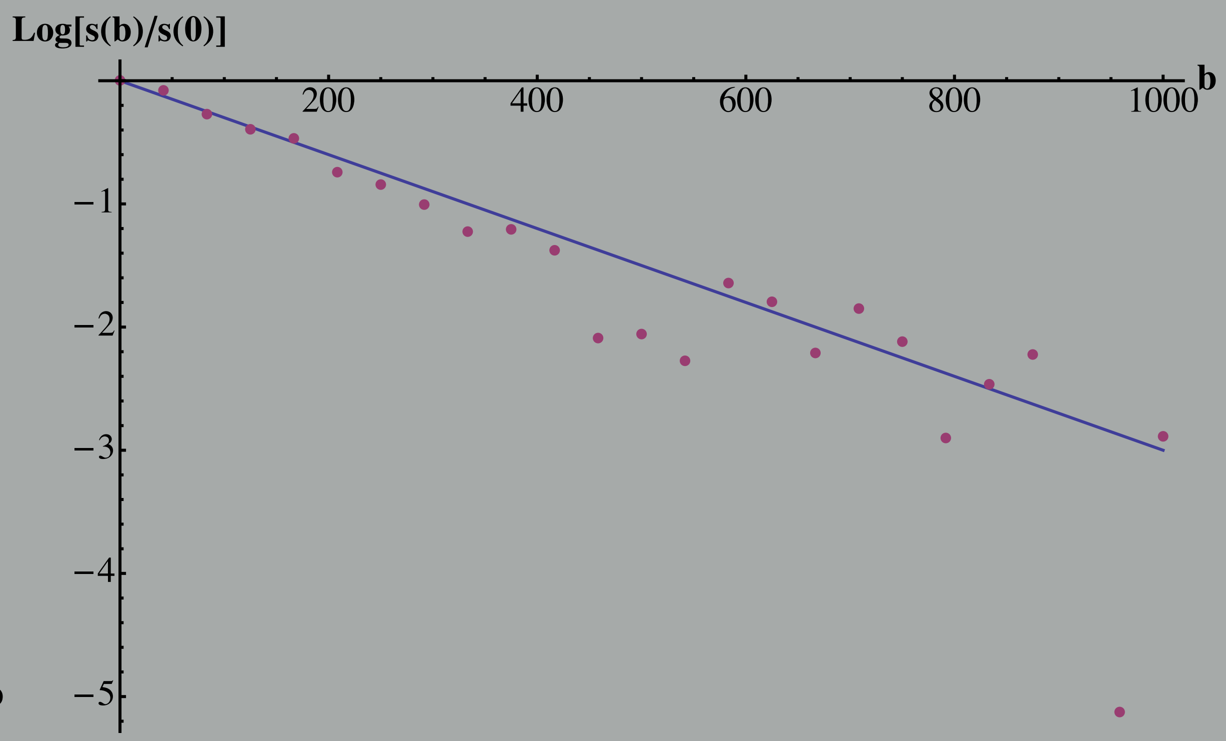


The signal from 1D Gaussian Diffusion

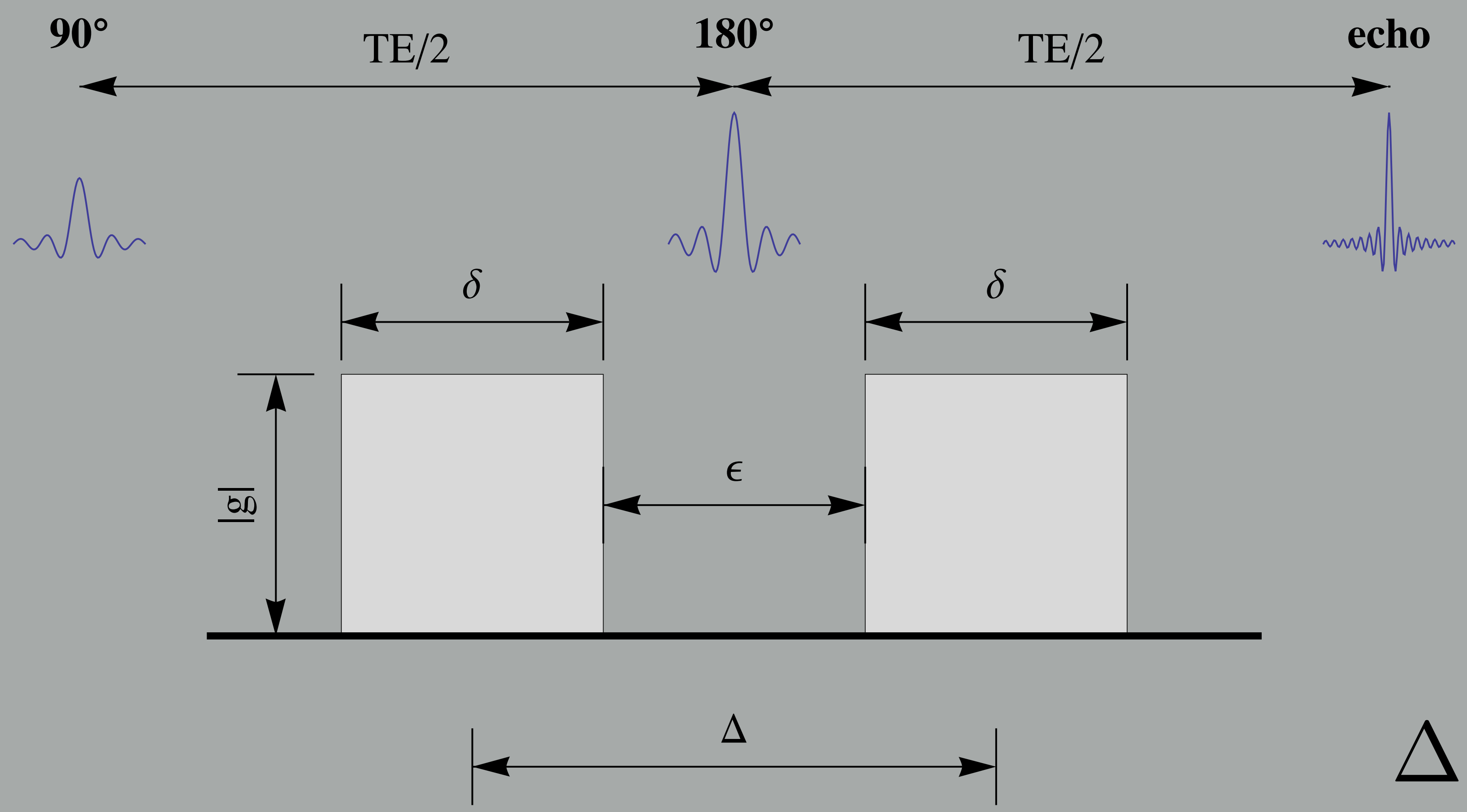
$$s(b) = s(0)e^{-bD} + \eta(b)$$

$$\log\left[\frac{s(b)}{s(0)}\right] \approx -bD$$





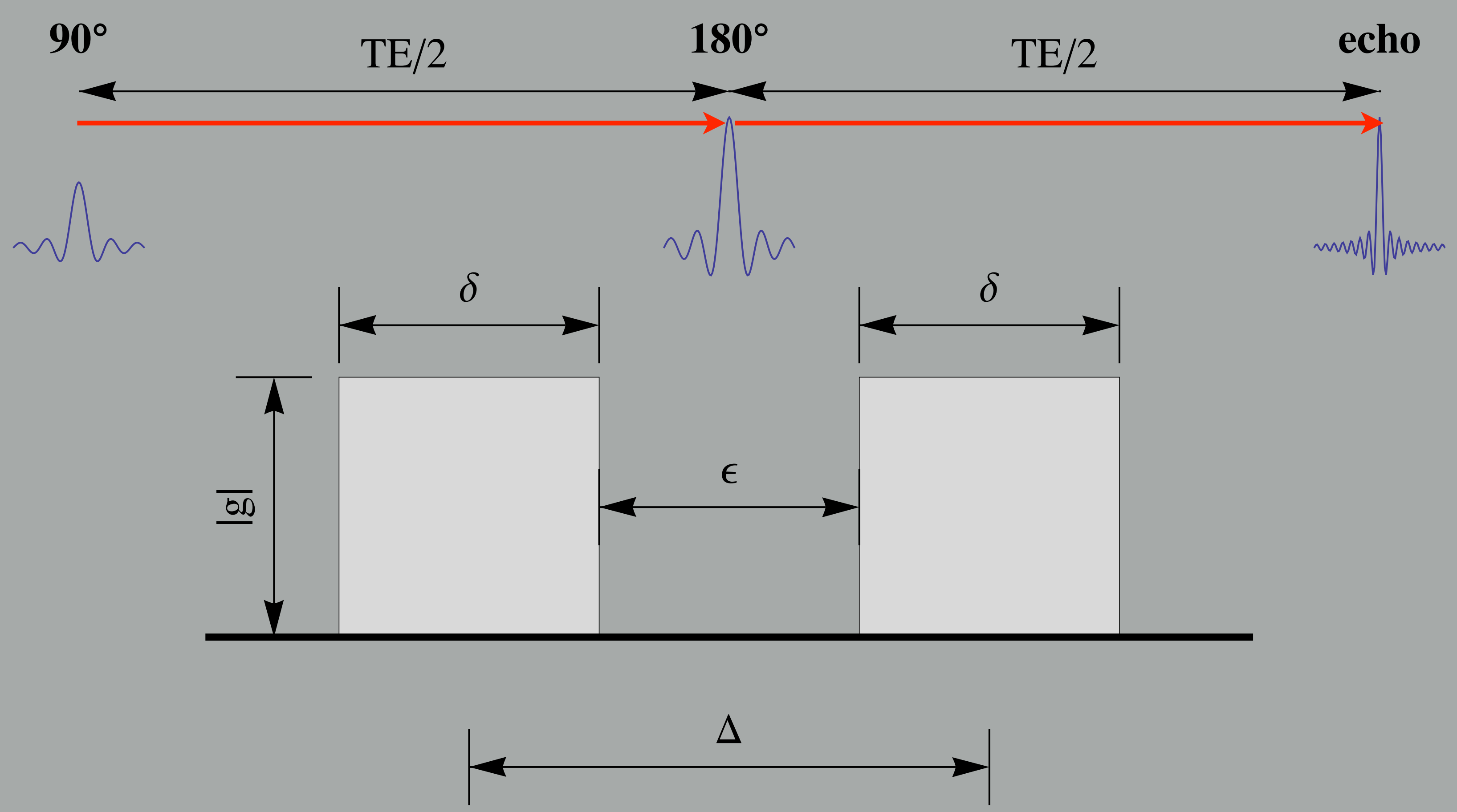
A “Standard” Bipolar Pulse



$$\Delta = \delta + \epsilon$$

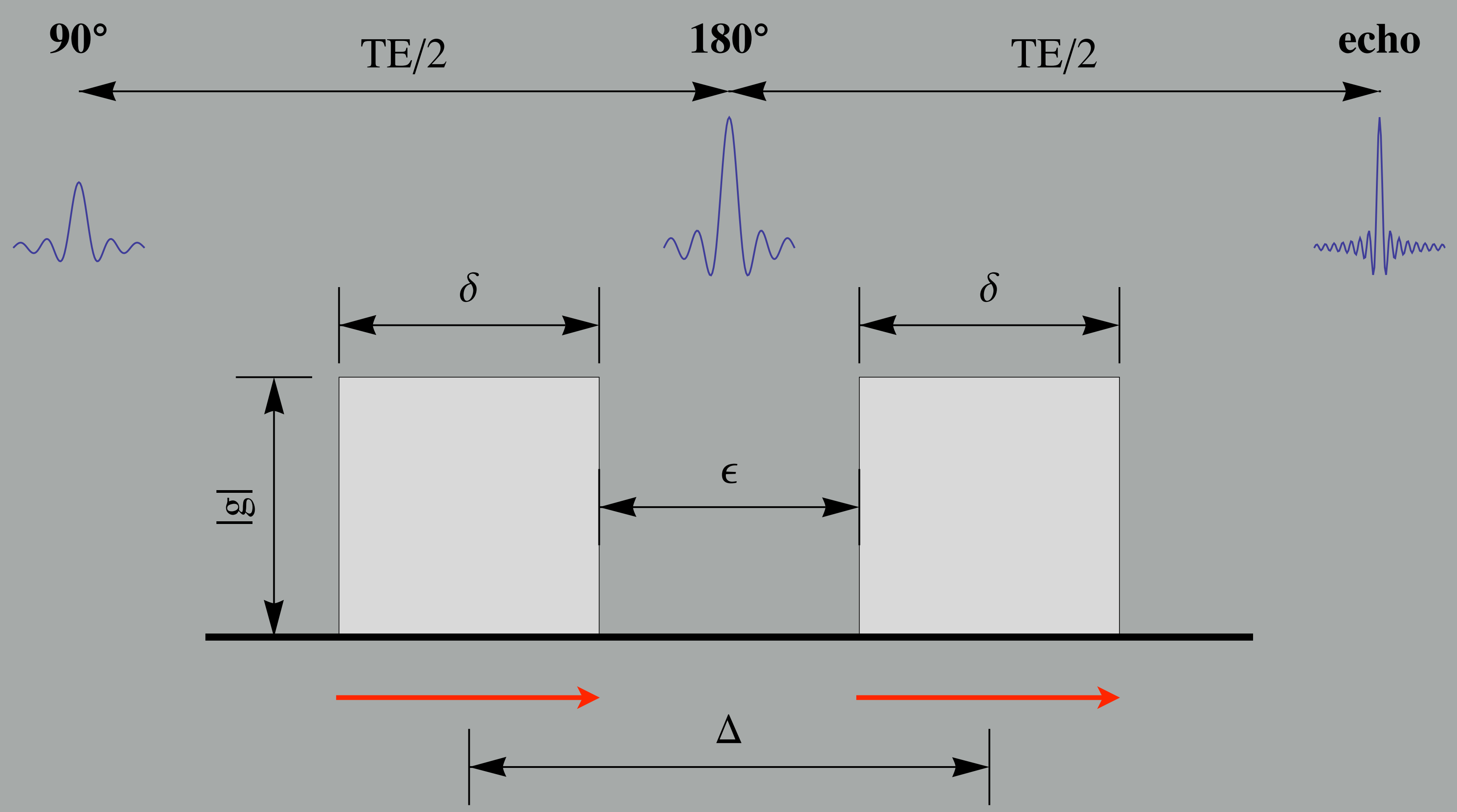
$$\varphi(\tau) = \int_0^\tau G(t)x(t)\,dt$$

A “Standard” Bipolar Pulse



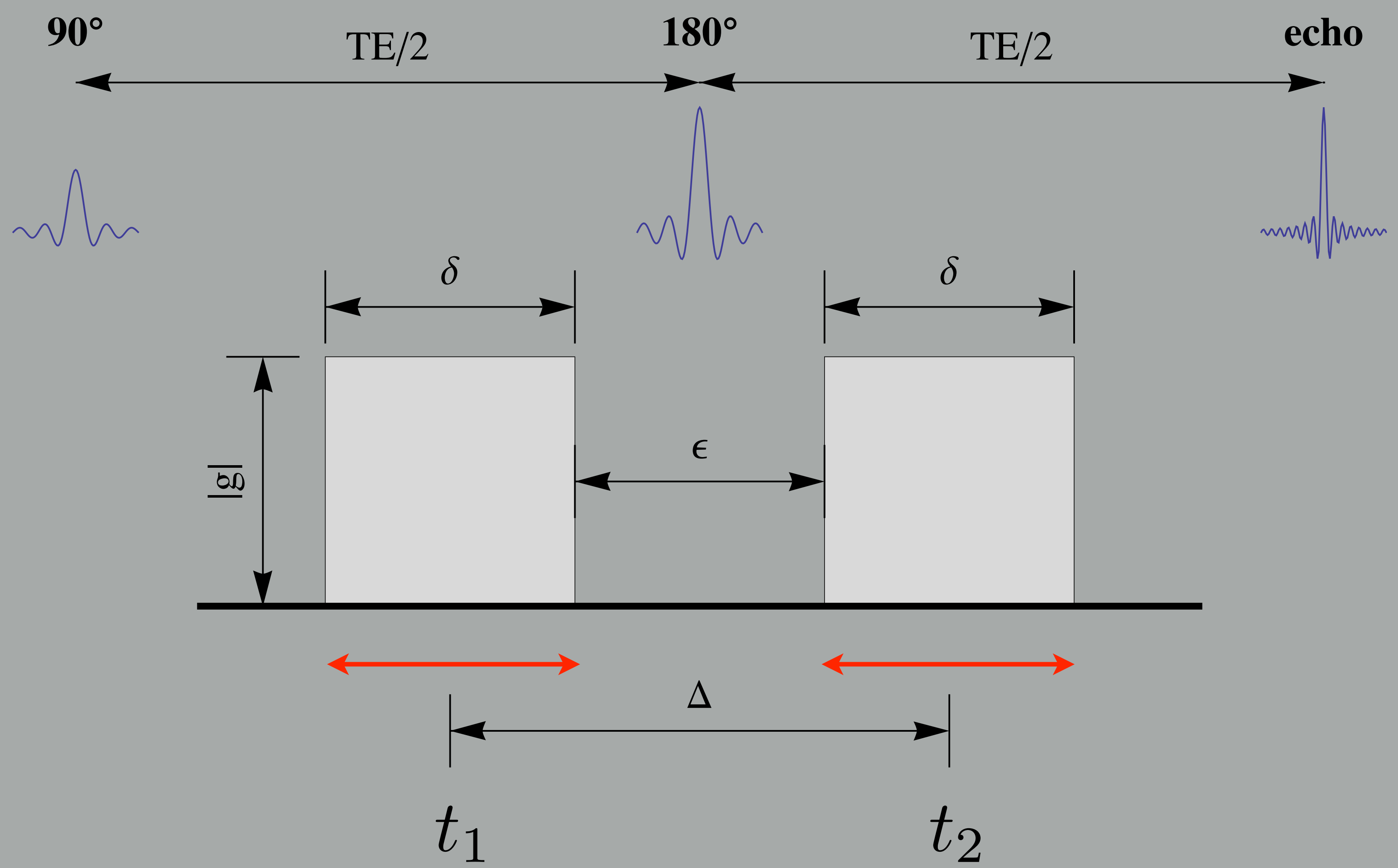
$$\varphi = -\int_0^{T_e/2} G(t)x(t)\ dt + \int_{T_e/2}^{T_e} G(t)x(t)\ dt$$

A "Standard" Bipolar Pulse



$$\varphi = -G\int_0^{\delta} x(t)\ dt + G\int_{\delta+\epsilon}^{2\delta+\epsilon} x(t)\ dt$$

A "Standard" Bipolar Pulse

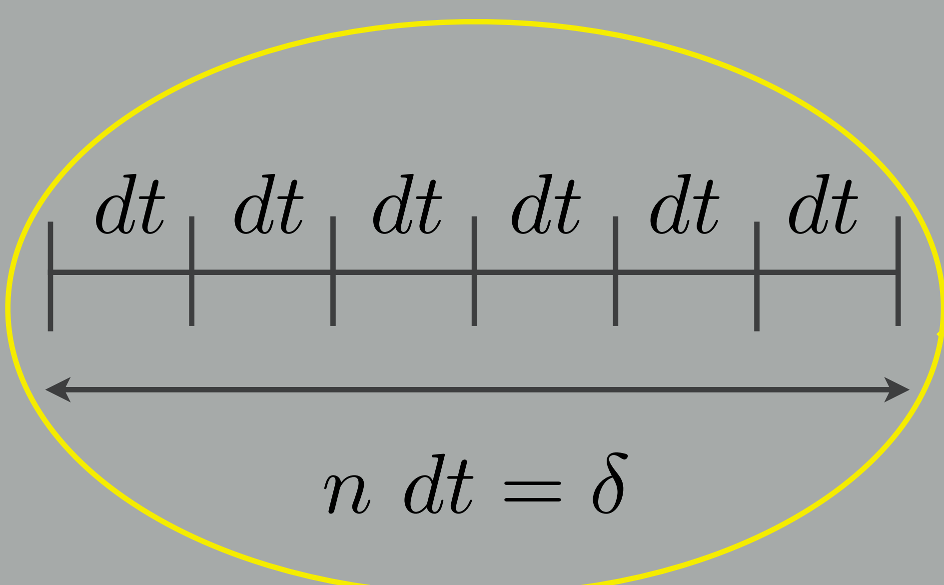


$$\varphi = -G \int_{-\delta/2}^{\delta/2} x(t)\ dt + G \int_{\Delta-\delta/2}^{\Delta+\delta/2} x(t)\ dt$$

The average location over time

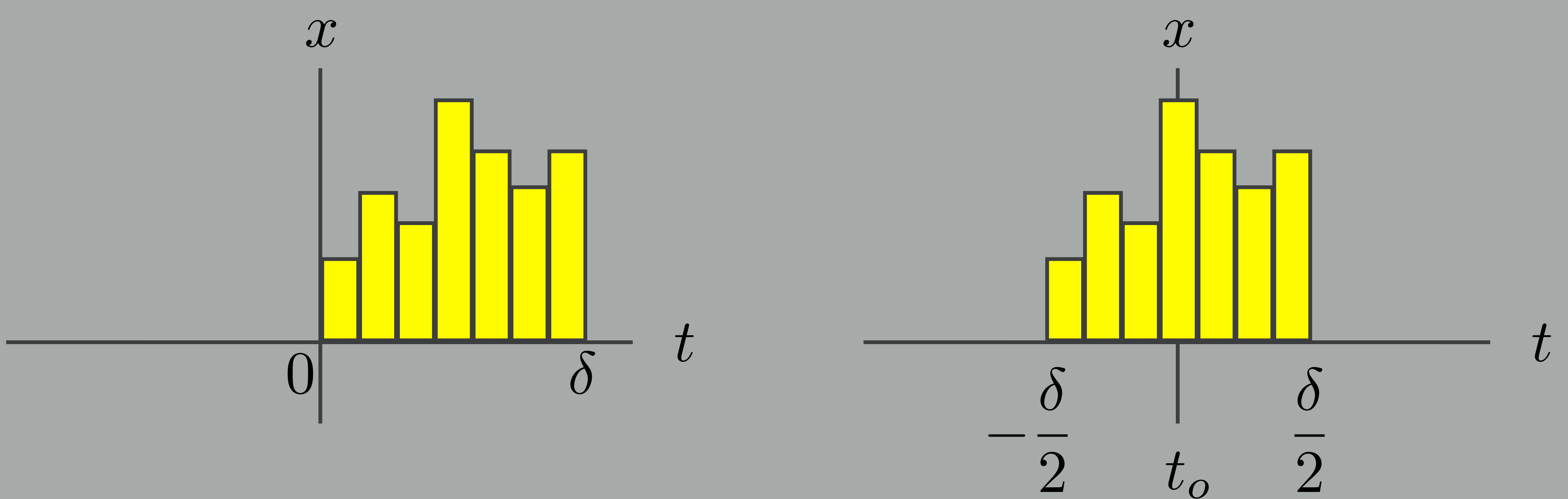
$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\bar{x} = \frac{1}{n\ dt} \sum_{i=1}^{n} x_i\ dt$$



$$\bar{x} = \frac{1}{\delta} \int_0^{\delta} x(t)\ dt$$

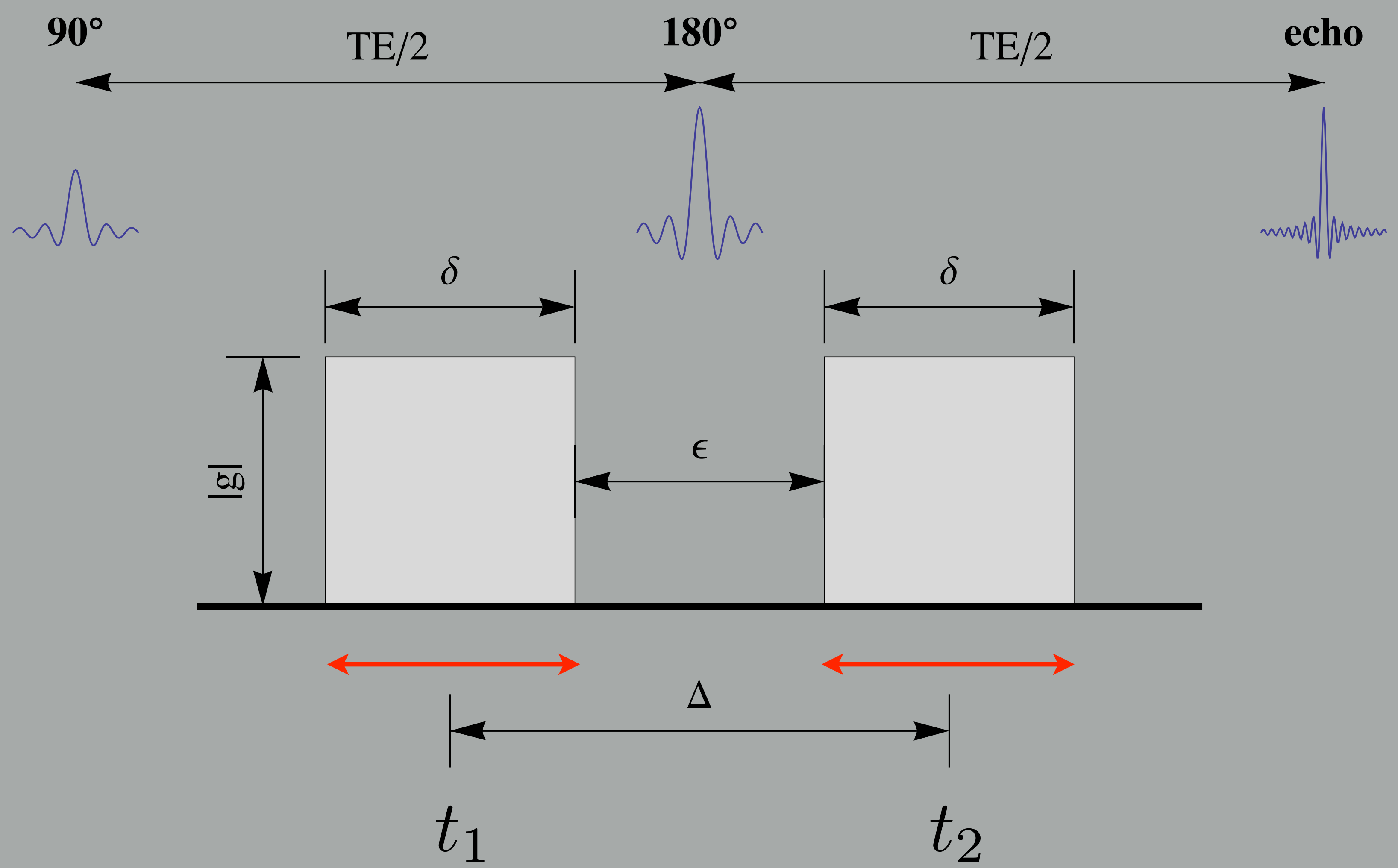
The average location over time



$$\bar{x} = \frac{1}{\delta}\int_0^{\delta} x(t)\ dt$$

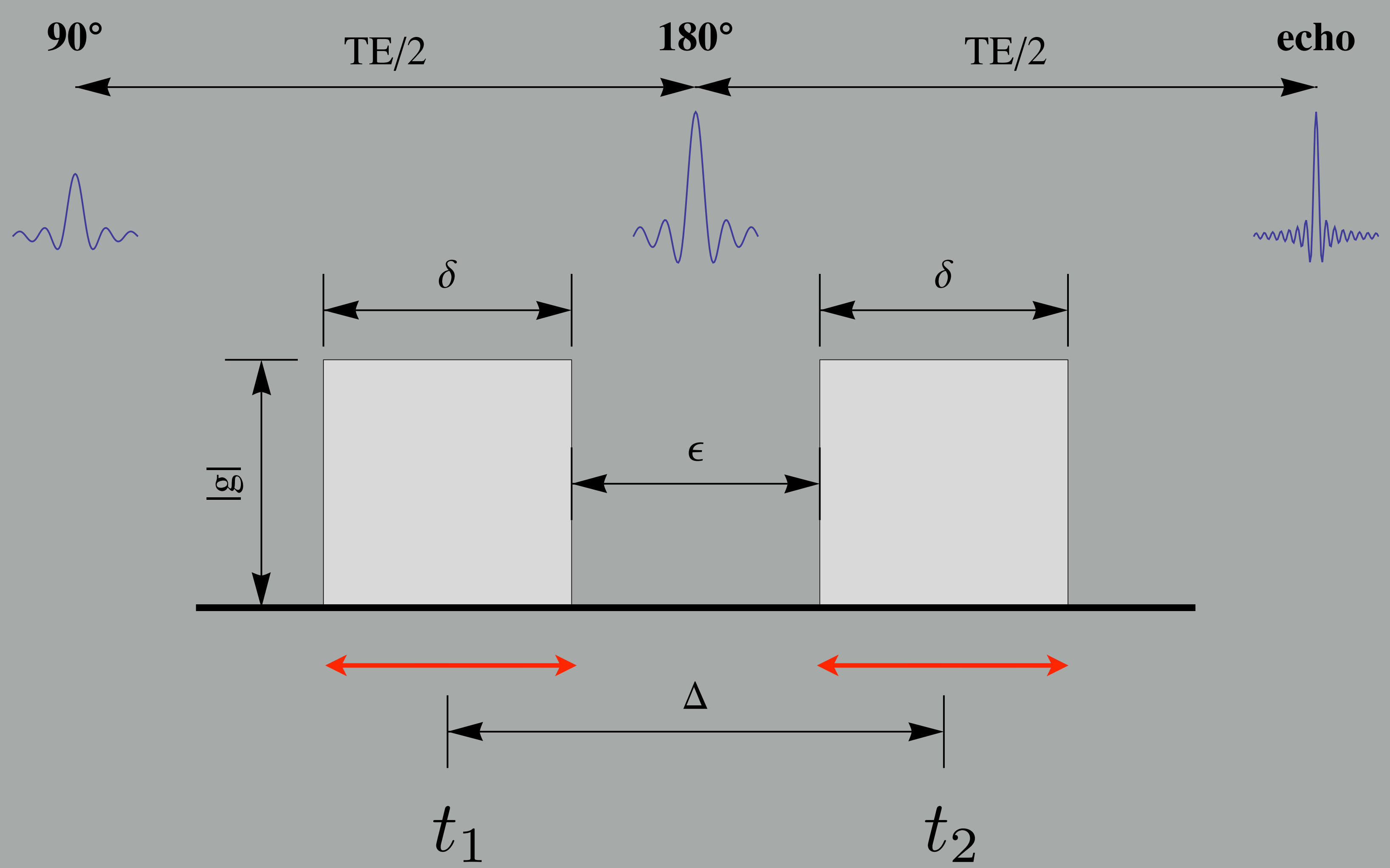
$$\bar{x}(t_o) = \frac{1}{\delta}\int_{t_o-\delta/2}^{t_o+\delta/2} x(t)\ dt$$

A "Standard" Bipolar Pulse



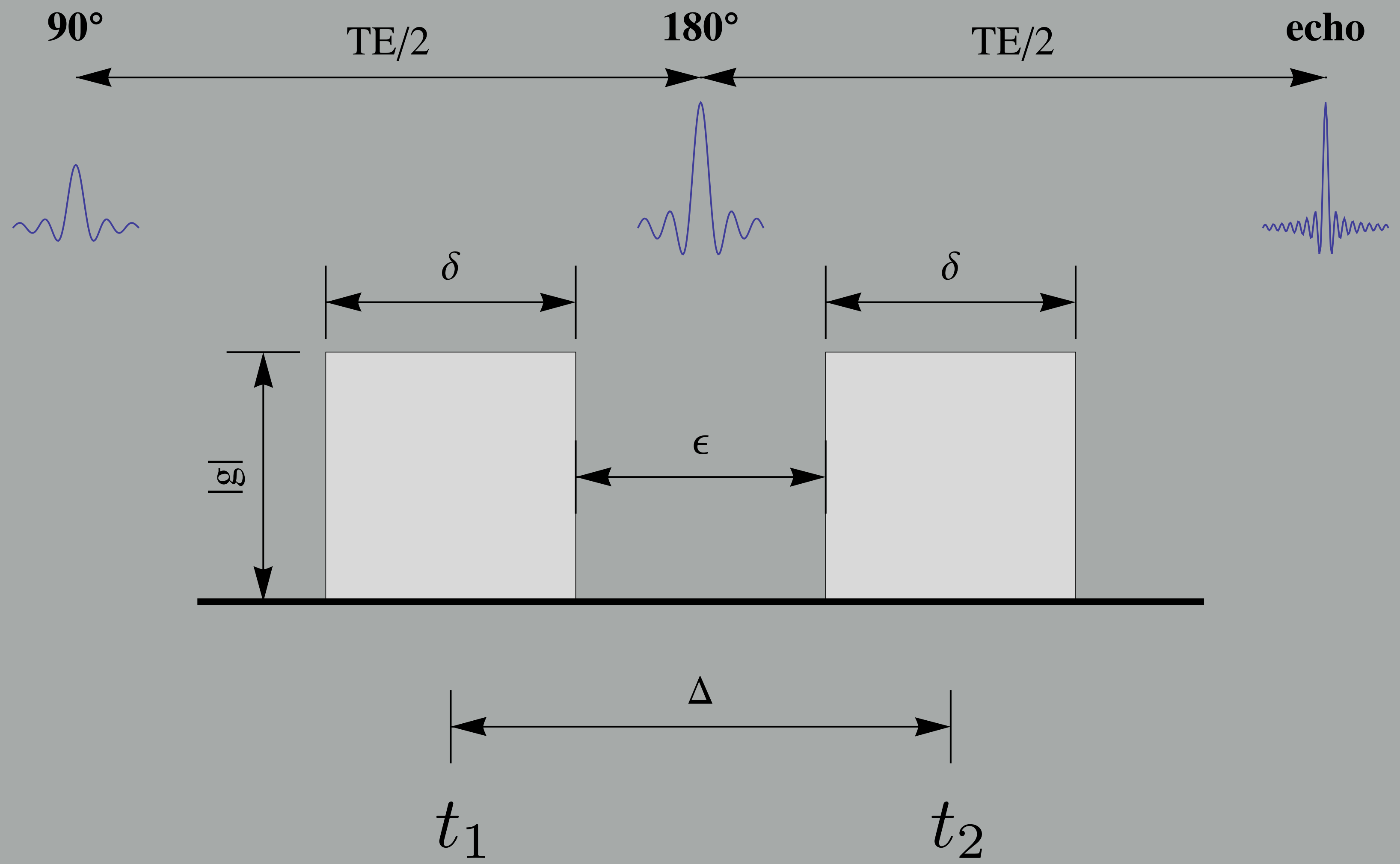
$$\varphi = -G\int_{t_1-\delta/2}^{t_1+\delta/2} x(t)\ dt + G\int_{t_2-\delta/2}^{t_2+\delta/2} x(t)\ dt$$

A "Standard" Bipolar Pulse



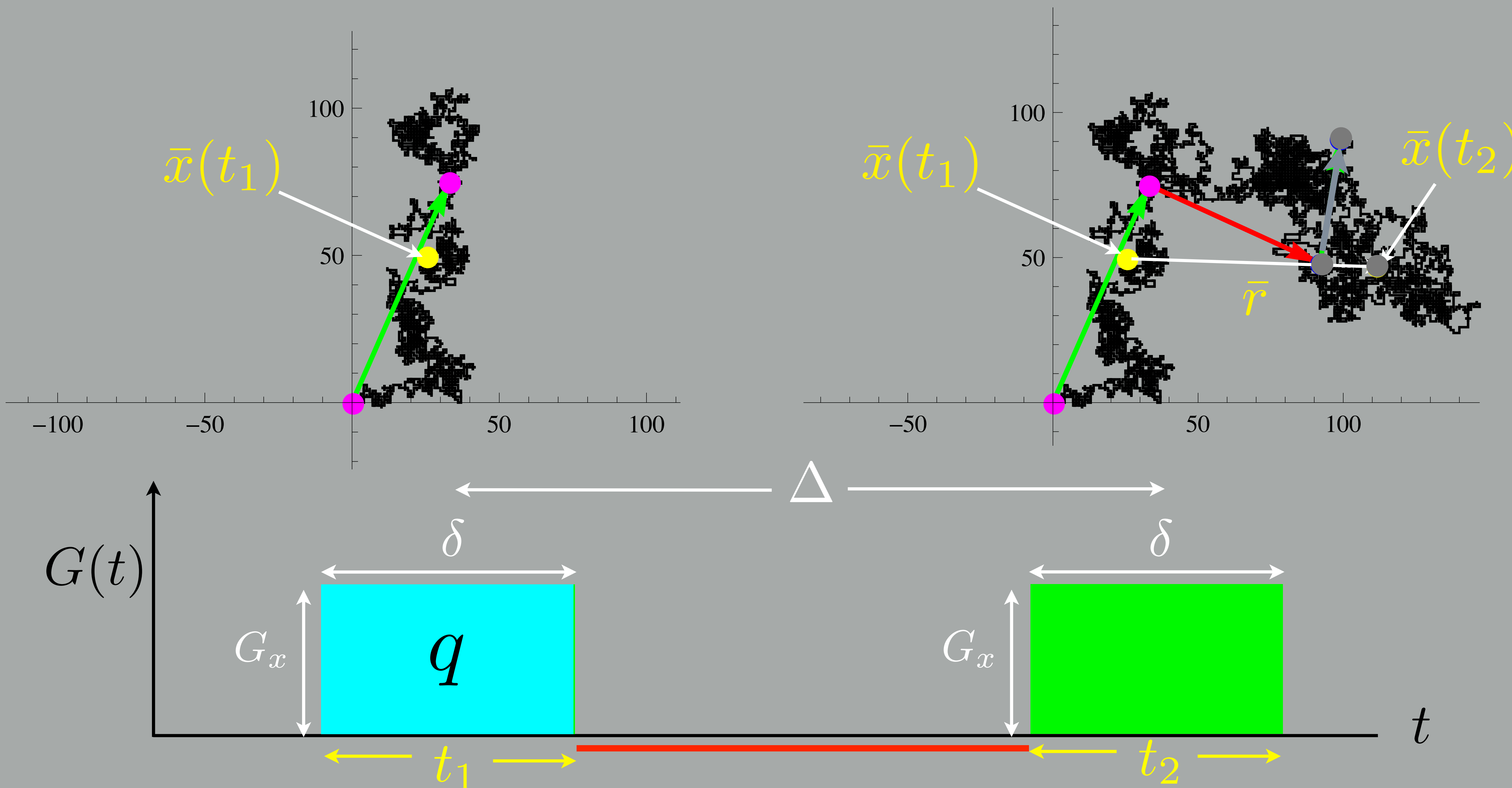
$$\varphi = \underbrace{-G\int_{t_1-\delta/2}^{t_1+\delta/2} x(t)\ dt}_{\delta\,\bar{x}(t_1)} + \underbrace{G\int_{t_2-\delta/2}^{t_2+\delta/2} x(t)\ dt}_{\delta\,\bar{x}(t_2)}$$

A "Standard" Bipolar Pulse



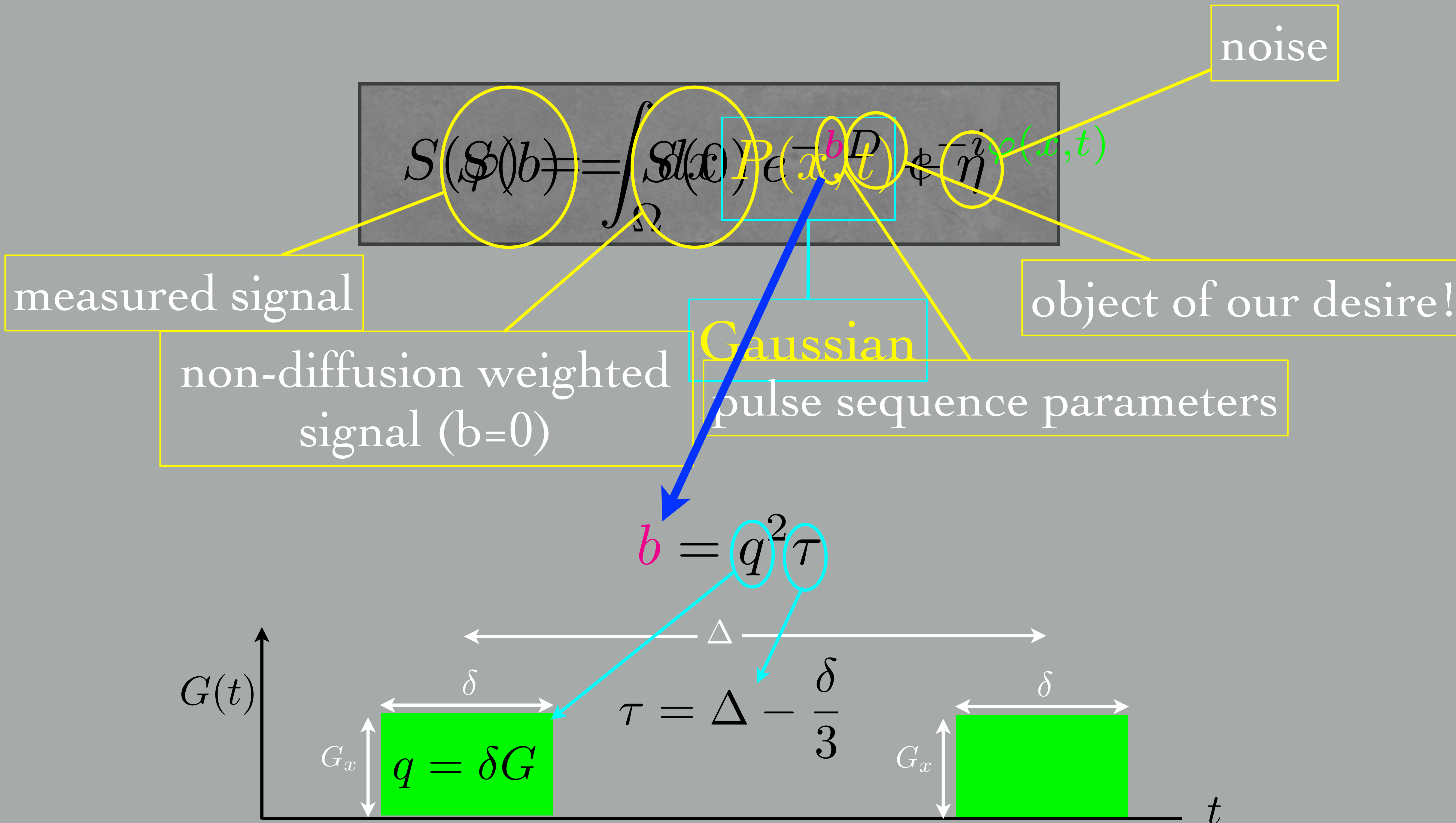
$$\varphi = \underbrace{G\delta}_{q}\ \underbrace{[\bar{x}(t_2) - \bar{x}(t_1)]}_{\bar{r}} = q\bar{r}$$

Diffusion phase in a Bipolar Pulse



$$\varphi(x,t) = \underbrace{G\delta}_{q}\,\underbrace{\left[\bar{x}(t_2) - \bar{x}(t_1)\right]}_{\bar{r}} = q\bar{r}$$

The Estimation Problem for Gaussian Diffusion



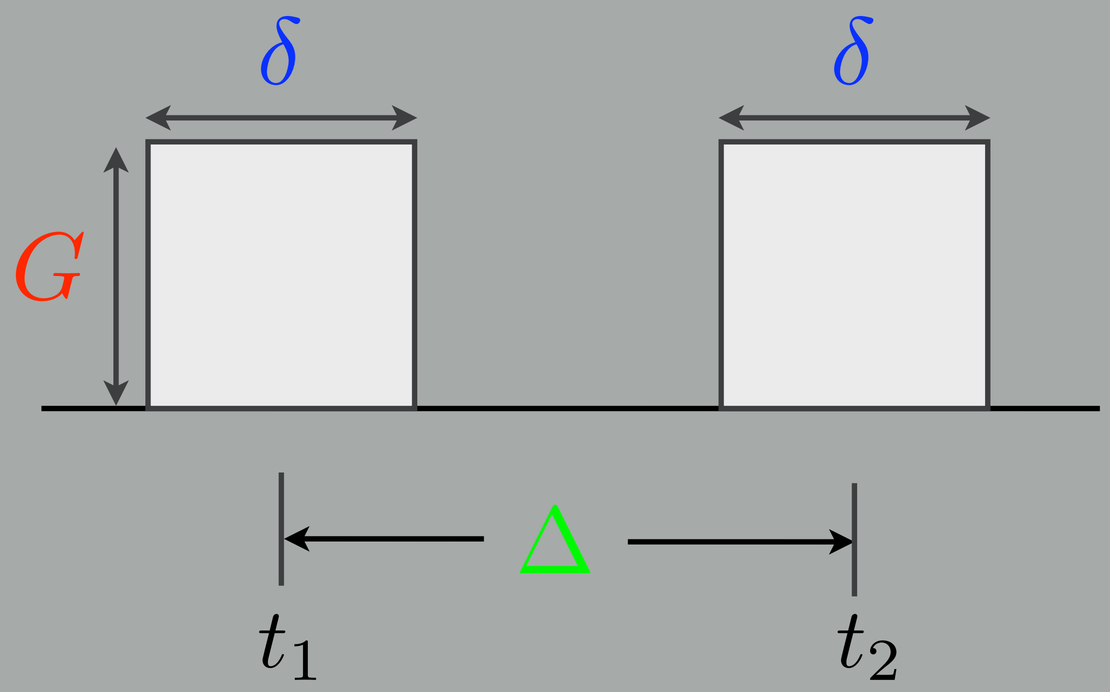
The NMR signal for 1D Gaussian diffusion

$$s(q,\tau) = s(0)\int P(\bar{r},\tau)e^{-iq\cdot\bar{r}}d\bar{r}$$

$$P(\bar{r},\tau) = \frac{1}{\sqrt{4\pi D\tau}}e^{-\bar{r}^2/(4D\tau)}$$

$$\boxed{s(q,\tau) = s(0)e^{-bD}}$$

The b-value



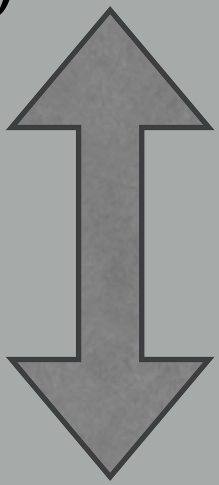
$$\Delta = t_2 - t_1$$

$$b = \underbrace{G^2\delta^2}_{q^2}\underbrace{(\Delta - \delta/3)}_{\tau}$$

The Diffusion Signal

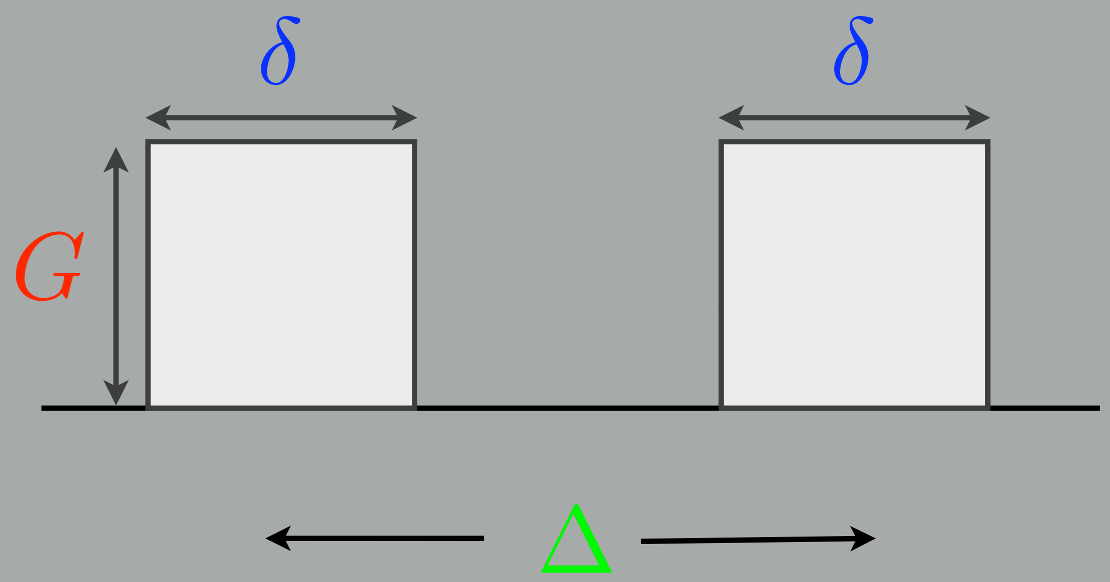
Signal and Distribution are
Fourier Transform pairs

$$\mathfrak{s}(q,\tau) = \int P(\bar{r},\tau)e^{-iq\cdot\bar{r}}d\bar{r}$$



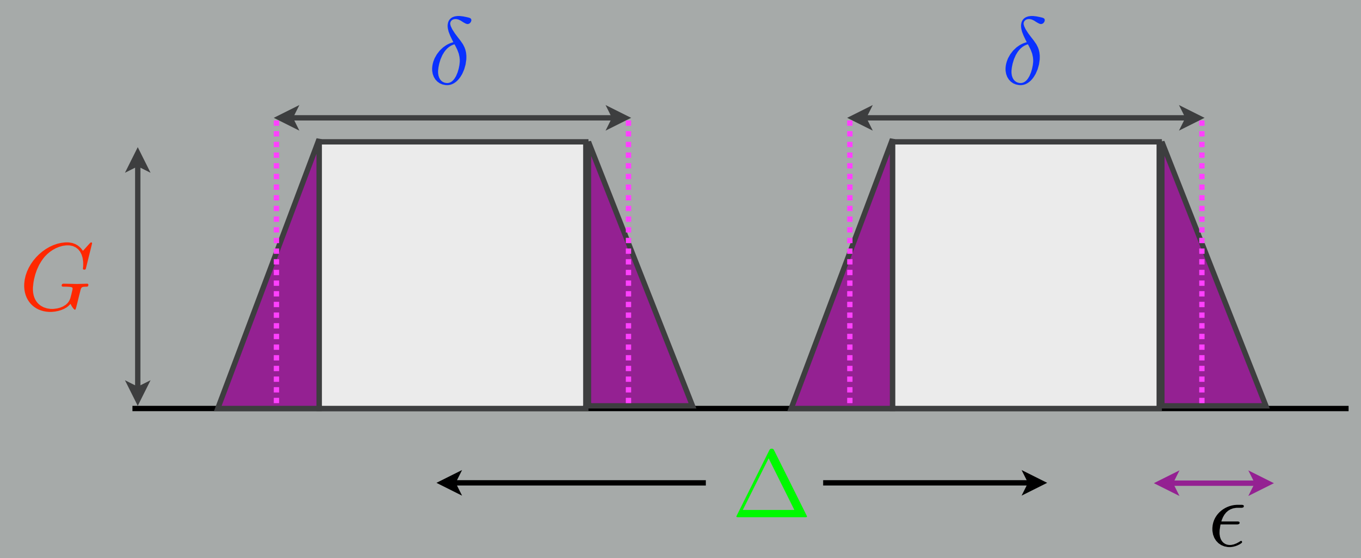
$$P(\bar{r},\tau) = \int \mathfrak{s}(q,\tau)e^{iq\cdot\bar{r}}dq$$

Ideal b-value



$$b = \underbrace{G^2\delta^2}_{q^2}\underbrace{(\Delta - \delta/3)}_{\tau}$$

Ideal b-value with trapezoidal pulses



$$b = G^2\left[\delta^2\tau - f(\delta,\epsilon)\right]$$

$$f(\delta,\epsilon) = \frac{1}{6}\delta\epsilon^2 - \frac{1}{30}\epsilon^3$$

The Diffusion Signal

Signal and Distribution are
Fourier Transform pairs

$$\mathfrak{s}(\boldsymbol{q}, \tau) = \int P(\bar{\boldsymbol{r}}, \tau) e^{-i\boldsymbol{q}\cdot\bar{\boldsymbol{r}}} d\bar{\boldsymbol{r}}$$

$$P(\bar{\boldsymbol{r}}, \tau) = \int \mathfrak{s}(\boldsymbol{q}, \tau) e^{i\boldsymbol{q}\cdot\bar{\boldsymbol{r}}} dq$$

Lecture Summary

1. Gradients and phase of stationary spins
2. Gradients and phase of spins at constant velocity
3. Gradients and phase of randomly moving spins