14 The Classical Description of NMR

14.1 Introduction

In the previous section we showed how the mysterious quantum mechanical property of *spin* possessed by the hydrogen nuclei (e.g., protons) in tissue water resulted in a macroscopic net or *bulk* magnetization $\boldsymbol{M}$ that is small but measureable when a large number of spins were placed in a large, static magnetic field $\boldsymbol{B}_o$. This magnetization points in the same direction as $\boldsymbol{B}_o$, which is usually defined to be the $\hat{\boldsymbol{z}}$ direction, and there is no component in the $x-y$ plane, because these "transverse" components of the many spins point in random directions, and thus average to zero. Moreover, the way this magnetization interacts with an *arbitrary*, time-varying external magnetic field $\boldsymbol{B}(t)$, its so-called *equation of motion*, can be described in completely *classical* terms, i.e., one need not know quantum mechanics. The equation of motion for the magnetization is the simple equation

$$\frac{d\boldsymbol{M}}{dt} = \gamma \boldsymbol{M} \times \boldsymbol{B} \tag{14.1}$$

where $\times$ is the cross product discussed in Section 3.14. From our discussion of the cross product in Section **??**, we can write this in the form of a matrix equation (dropping the explicit time dependence for simplicity):

$$\frac{d\boldsymbol{M}}{dt} = \boldsymbol{\Omega}\boldsymbol{M} \tag{14.2}$$

where

$$\boldsymbol{M} = \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} \quad , \quad \boldsymbol{\Omega} = \gamma \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix} \tag{14.3}$$

It will be helpful to recall that the matrix form of the cross product is formed from the dot product of $\boldsymbol{B}$ with the component matrices Eqns **??**- **??** (called the *infinitesimal generators*) representing the cross product:

$$\boldsymbol{\Omega} = \gamma B_x \boldsymbol{A}_x + \gamma B_y \boldsymbol{A}_y + \gamma B_z \boldsymbol{A}_z \tag{14.4}$$

This expression is very useful for investigating different components of the applied magnetic fields.

In addition to the motion of the magnetization due to the applied magnetic fields, we also saw that the order caused by the immersion of the spins in a large static field becomes disordered over time, in a process called *relaxation*. The spins interact with their environment so that the longitudinal component of magnetization comes to equilibrium through an exponential process governed by the time constant T_1. The spins interact with each other, causing an exponential decay in the coherence of the transverse magnetization through an exponential process governed by the time constant T_2. The key equation for bulk measurements in NMR is thus the equation that combines the motion of the bulk magnetization in the presence of external magnetic fields (Eqn 14.1) with the process of relaxation. What we will show momentarily is that relaxation effects are very easily incorporated into Eqn 14.2 by the addition of the product of another simple matrix, the *relaxation rate matrix* $\boldsymbol{R}$, with $\boldsymbol{M}$:

$$\frac{d\boldsymbol{M}}{dt} = \boldsymbol{\Omega M} + \boldsymbol{RM} \tag{14.5}$$

where

$$\boldsymbol{R} = \begin{pmatrix} 1/T_2 & 0 & 0 \\ 0 & 1/T_2 & 0 \\ 0 & 0 & 1/T_1 \end{pmatrix} \tag{14.6}$$

This nomenclature *rate* derives from the fact that the inverse of the relaxation times, $\{R_1, R_2\} \equiv \{1/T_1, 1/T_2\}$, are called *relaxation rates*. Now we only have one more thing to do to make Eqn 14.5 a legitimate equation describing the magnetization. Imagine that we turn off the external fields (i.e., $\boldsymbol{\Omega} = 0$) and let the sample sit in the magnet. Eventually, it will come to equilibrium with the main field, meaning it will stop changing with time: $d\boldsymbol{M}/dt = 0$ and will have some equilibrium value which we will call M_{eq}. We thus see that for Eqn 14.5 to be physically accurate, it must incorporate this *boundary condition* by the addition of an additional term:

$$\frac{d\boldsymbol{M}}{dt} + [\boldsymbol{\Omega} + \boldsymbol{R}]\,\boldsymbol{M} = \boldsymbol{W} \qquad \text{Bloch Equation} \tag{14.7}$$

where $\boldsymbol{W} = \boldsymbol{R M}_{eq}$. With the standard convention that the main field is aligned along the $\hat{\boldsymbol{z}}$ axis, the equilibrium magnetization is

$$\boldsymbol{M}_{eq} = \begin{pmatrix} 0 \\ 0 \\ M_o \end{pmatrix} \tag{14.8}$$

The magnetization is aligned along $\hat{\boldsymbol{z}}$ and there is no net transverse magnetization.

Eqn 14.7 is called the *Bloch Equation* (**?**). It tells us how the magnetization $\boldsymbol{M}(t)$ evolves in time in the presence of an applied magnetic field $\boldsymbol{B}(t)$ and relaxation time constants T_1 and T_2. Solving the Bloch equation for a given field $\boldsymbol{B}(t)$ tells us $\boldsymbol{M}(t)$, that is, how the magnetization evolves with time. This equation is central to MRI and understanding it is critical to getting a grasp of basic imaging strategies. In what follows, we will break it down in such a way to highlight some key features, eliminate its mystery, and show how some of the important features of constrast in MRI can be understood, before we ever get to imaging!

14.2 Free Precession

Let us first consider the case in which spins are immersed in only the large static field $B_z = B_o\hat{z}$. Let us also neglect relaxation, which means that we assume $\{T_1, T_2\} = \{\infty, \infty\}$. The Bloch equation Eqn 14.7 simplifies to

$$\frac{d\boldsymbol{M}}{dt} = \boldsymbol{\Omega}_0\boldsymbol{M} \tag{14.9}$$

where

$$\mathbf{\Omega}_o = \begin{pmatrix} 0 & -\omega_o & 0 \\ \omega_o & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14.10}$$

and

$$\omega_o = \gamma B_o \tag{14.11}$$

is called the *Larmor frequency*. In component form this is simply

$$\begin{pmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{pmatrix} = \begin{pmatrix} 0 & -\omega_o & 0 \\ \omega_o & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} \tag{14.12}$$

Writing out the components this is

$$\frac{dM_x}{dt} = -\omega_o M_y \tag{14.13a}$$

$$\frac{dM_y}{dt} = \omega_o M_x \tag{14.13b}$$

Taking a hint from our discussion of complex numbers (Chapter 4) and defining the *transverse magnetization* [1]

$$M_{xy} = M_x + iM_y \tag{14.14}$$

we can write Eqn 14.13 as a single complex equation:

$$\frac{dM_{xy}}{dt} = -i\omega_o M_{xy} \tag{14.15}$$

which can be easily solved:

$$M_{xy}(t) = M_{xy}(0)e^{-i\omega_o t} \tag{14.16}$$

Eqn 14.16 says that the transverse magnetization *precesses* about the $\hat{z}$ axis with frequency ω_o. This is shown in Figure 14.1. It is just the macroscopic equivalent of the results of Section 13.4. Please keep in mind that we are referring to the *bulk* magnetization here. As discussed in Chapter 13, the individual spins are *always* precessing in the static magnetic field.

This simplest of situations (we've only put a sample in the main field, and nothing else!) has led us to two very important concepts. The first is that there is an advantage to describing the magnetization in terms of transverse and longitudinal components $\{M_x + iM_y, M_z\}$, rather than three cartesian components $\{M_x, M_y, M_z\}$, and that the complex representation facilitated this description. In a later discussion, we will extend this complex representation to the general Bloch equation.

The second concept is implicit in the description of Eqn 14.16 of the motion of the magnetization in the main field as a vector rotating at the Larmor (angular) frequency ω_o, which brings us to the concept of the rotating frame of reference.

[1] figure of $\{M_x, M_y, M_z\}$ and $\{M_{xy}, M_z\}$.

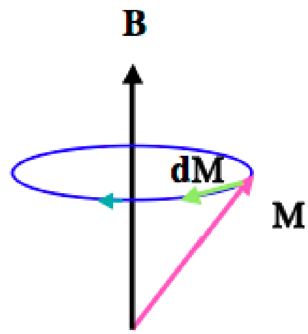


Figure 14.1 Free precession of the magnetization about the main field.

14.3 The rotating frame of reference

The main magnetic field $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$ is always on and so the precession described in the last section is always occurring and will be so during whatever other complicated manipulations of the fields we perform. [2] In other words, our experiments can be thought of as taking place in a *reference frame that rotates at the Larmor frequency*, which we discussed in Section **??**. For the most part MRI experiments are thought of in this frame and their mathematical description is greatly simplified by transforming to this frame.

But we have actually already seen this exact problem in Section 6.12 and know the answer (Eqn 6.61):

$$\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} = \left(\frac{d\boldsymbol{M}}{dt}\right)_{lab} + \boldsymbol{\Omega}_r\boldsymbol{M} \tag{14.17}$$

where $\boldsymbol{\Omega}_r$ is the reference $\boldsymbol{\Omega}$ derived from the magnetic fields what we want to constitute our reference frame. Transforming the general form of the Bloch equations Eqn 14.5 to any reference frame using Eqn 14.17 gives

$$\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} = (\boldsymbol{\Omega} + \boldsymbol{\Omega}_r)\boldsymbol{M} + \boldsymbol{R}\boldsymbol{M} \tag{14.18}$$

which is equivalent to the Bloch equation for an *effective magnetic field* $\boldsymbol{B}_e$ seen by the magnetization in the new reference frame

$$\boldsymbol{B}_e = \boldsymbol{\Omega}_e/\gamma = (\boldsymbol{\Omega} + \boldsymbol{\Omega}_r)/\gamma \tag{14.19}$$

That is, *in the rotating frame, the magnetization precesses about the effective field* B_e. For example, in the previous example of free precession, if we choose the reference frame as

$$\boldsymbol{\Omega}_r = -\boldsymbol{\Omega}_0 = \begin{pmatrix} 0 & \omega_0 & 0 \\ -\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14.20}$$

[2] It is worth recalling at this point that this frequency depends upon the gyromagnetic ratio γ, and so is dependent upon the chemical species. But for MRI we are almost always talking about a single species - water, and so a single γ.

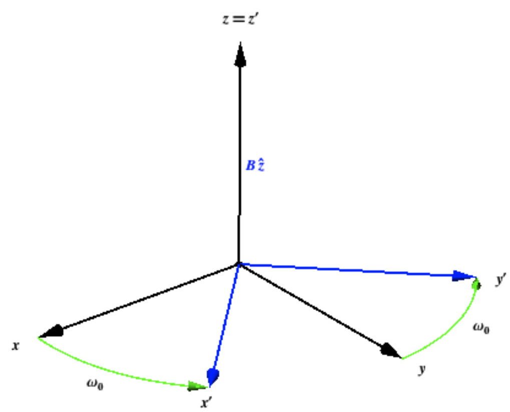


Figure 14.2 The rotating frame (blue) rotates at angular frequency ω_0 relative to the laboratory frame (black) about the $\hat{z}$-axis defined by the direction of the main (static) field $B_z = B_o\hat{z}$. (**direction of rotation?**)

and neglect relaxation ($\boldsymbol{R} = \mathbf{0}$), on the grounds that the relaxation time are much longer than the pulse duration ($\tau \ll T_1, T_2$), we see that

$$\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} = 0 \tag{14.21}$$

which says that the magnetization vector in this frame sees *no* effective field, and thus is *stationary* (i.e., does not change with time) in the rotating frame for free precession in the absence of relaxation. This particular rotating frame is called the *Larmor rotating frame* because its angular frequency is the Larmor frequency. This frame has particular significance in MRI because it is the frame associated with the main (static) field $\boldsymbol{B}_0$, which is always on. Using this frame in calculations thus allows us to ignore the effects of the main field, which greatly simplifies the equations. The laboratory and rotating frames are shown in Figure 14.2. We can transform the general form of the Bloch equations Eqn 14.5 to this reference frame using Eqn 14.17:

$$\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} = (\boldsymbol{\Omega} - \boldsymbol{\Omega}_o)\boldsymbol{M} + \boldsymbol{R}\boldsymbol{M} \tag{14.22}$$

It is this form of the Bloch equations that is most often used in the description of MRI.

A comment on notation is worth mentioning here. It is common in the literature to describe the lab coordinates with the Cartesian axes $\{x, y, z\}$ and those in the rotating system as $\{x', y', z'\}$. This is useful in an initial discussion of the rotating frame. But in MRI we are almost always working in the rotating frame, since the subject is always in a large static field $B_o\hat{\boldsymbol{z}}$. So the "primed" notation used to distinguish the lab frame axis x from the rotating frame axis x' would require that we carry around those primes everywhere. Since we will be working almost exclusively in the rotating frame, however, we follow the convention of dropping the primes and adopting the implicit assumption that we are working in the rotating frame unless otherwise specified.

Problems

14.1 In what way is the effective field in the rotating frame analogous to the Coriolis effect that causes large scale storm systems to veer to the right in the Northern Hemisphere?

14.4 The Bloch Equations in Complex Form: The Axial Representation

The transverse components of the magnetization $\{m_x, m_y\}$ and the longitudinal component m_z clearly have different status experimentally. The transverse components are detected by the coil, and decay according to T_2, whereas the longitudinal component is not detected and recovers with T_1. Now, we have seen that the motion of the magnetization vector in the static field is a precession about $\hat{z}$, and we also know from our discussion of complex numbers (Chapter 4) that rotations in a plane are succinctly described by writing the two coordinates $\{x, y\}$ as a single complex coordinate $x + iy$. Therefore describing the magnetization explicitly in terms of transverse and longitudinal components is a very useful representation. This is called the *axial representation*:

$$\boldsymbol{M}_{cartesian} = \begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} \tag{14.23a}$$

$$\boldsymbol{M}_{axial} = \begin{pmatrix} M_{xy}^{+} \\ M_{xy}^{-} \\ M_z \end{pmatrix} = \begin{pmatrix} M_x + iM_y \\ M_x - iM_y \\ M_z \end{pmatrix} \tag{14.23b}$$

which now represents the magnetization in terms of the longitudinal component M_z and two components in the transverse plane, one rotating counter-clockwise $M_{xy}^{+} \equiv M_x + iM_y$, and one rotating clockwise $M_{xy}^{-} \equiv M_x - iM_y = M_{xy}^{+*}$. Note that this representation involves *two* transverse components that are complex conjugates (Chapter 4) of one another: $M_{xy}^{+} = M_{xy}^{-*}$, and thus represent transverse components rotating in opposite directions [3]. The significance of this will be evident in our discussion of coherence pathways in Chapter 21: they allow us to represent magnetization components both coming into phase and going out of phase with RF pulses.

The transformation from the Cartesian to the axial representation is achieved with the matrix (**?**)

$$\boldsymbol{P} = \begin{pmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14.24}$$

So that

$$\boldsymbol{M}_{axial} = \boldsymbol{P}\boldsymbol{M}_{Cartesian} \tag{14.25a}$$

$$\boldsymbol{\Omega}_{axial} = \boldsymbol{P}\boldsymbol{\Omega}_{Cartesian}\boldsymbol{P}^{-1} \tag{14.25b}$$

where we have used the transformation rules of Chapter **??**: the vector $\boldsymbol{M}$ is transformed by the dot product with $\boldsymbol{P}$ while the frequency matrix $\boldsymbol{\Omega}$ requires a similarity transformation (Section **??**).

[3] Sometimes these are written $M_{\perp} = M_x + iM_y$ and $M_{\parallel} = M_z$

Now, all we have done is change our representation of the magnetization. But as before, we still want to describe the effect of an RF pulse on the magnetization vector. However, since we've changed representation of the magnetization vector, we'll need to change our representation of the RF pulses so that in either representation the resulting magnetization vector does the same thing for the same pulse. This means that we need to figure out how to represent rotations in the axial representation. But note that the axial representation is somewhat strange in that the three orthogonal Cartesian axes $\{x, y, z\}$, have been combined into a plane (x and y), and the z-axis orthogonal to the $x - y$ plane. So, in some sense we have taken a 3-dimensional representation and transformed it into a 2-dimensional representation. In Eqn 21.2 we saw that the Cartesian coordinate system, rotations were described in terms of our familiar 3D cartesian rotation matrices $\boldsymbol{R}_c$ discussed in Section **??**. But finding the equivalent rotation in the axial representation turns out to be a non-trivial exercise. To see why, intuitively, this might be the case, consider the two major differences between the Cartesian and the axial representations: 1) The Cartesian has real components while the axial has complex components; 2) The Cartesian involved three dimensions while the axial has two dimensions. The first point is clear, but the second is more subtle: the magnetization vector is still moving in the 3-dimensional space of magnetization, but its *representation* involves only two dimensions: the longitudinal axis and the transverse plane.

Therefore, the task is to find the 3×3 rotation matrix $\boldsymbol{R}_a$ in the axial representation that converts the magnetization, in the axial representation, just before the pulse $\boldsymbol{M}_a^-$ to the magnetization immediately following the pulse $\boldsymbol{M}_a^+$:

$$\boldsymbol{M}_a^+ = \boldsymbol{R}_a \boldsymbol{M}_a^- \tag{14.26}$$

We'll drop the subscript a in what follows.

Consider a magnetization vector $\boldsymbol{M}$ pointing in an arbitrary direction described in terms of the spherical polar coordinates: the polar angle is θ, the azimuth angle is φ, and the radius $r = M \equiv \|\boldsymbol{M}\|$, which we will consider to be constant for now. It remains on the surface of what is called the *Bloch sphere*. [4]

$$\boldsymbol{M} = \begin{pmatrix} M_{xy} \\ M_{xy}^* \\ M_z \end{pmatrix} = \begin{pmatrix} M \sin\theta \, e^{i\varphi} \\ M \sin\theta \, e^{-i\varphi} \\ M \cos\theta \end{pmatrix} \tag{14.27}$$

Each such magnetization can be associated with a *spinor*

$$\boldsymbol{\psi} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} M^{1/2} \cos(\theta/2) \, e^{-i\varphi/2} \\ M^{1/2} \sin(\theta/2) \, e^{i\varphi/2} \end{pmatrix} \tag{14.28}$$

Thus the rotation of the 3-dimensional magnetization vector from $\boldsymbol{M}^-$ to $\boldsymbol{M}^+$ by the 3-dimensional rotation matrix $\boldsymbol{R}$ (Eqn 14.26) is now represented by a 2-dimensional complex unitary transformation of ψ:

$$\boldsymbol{\psi}^+ = \boldsymbol{U} \boldsymbol{\psi}^- \tag{14.29}$$

where $\boldsymbol{U}$ is a unitary matrix given in its general form by Eqn **??**.

[4] Put a figure here!

14.5 General Solution of the Bloch Equation

For $B_1 \ll B_0$, the solution to Eqn 14.7 can be found using the Caley-Hamilton theorem (**?**):

$$\boldsymbol{M}(t) = \boldsymbol{F}(t)\left[\boldsymbol{M}(0) - \boldsymbol{M}_{eq}\right] + \boldsymbol{M}_{eq} \tag{14.30}$$

where $\boldsymbol{M}(0)$ is the initial magnetization, $\boldsymbol{M}_{eq} = \boldsymbol{M}(t \to \infty)$ is the equilibrium magnetization, given by

$$\boldsymbol{M}_{eq} = \left(\boldsymbol{T}^{-1} + \boldsymbol{\Omega}\right)^{-1} \boldsymbol{W} \tag{14.31}$$

and $\boldsymbol{F}(t)$ is the function that takes, or propagates, the magnetization from its value at one time to a subsequent time, and is called the *propagator*:

$$\boldsymbol{F}(t) = \exp\left[-(\boldsymbol{T}^{-1} + \boldsymbol{\Omega})t\right] \tag{14.32}$$

The $\boldsymbol{F}(t)$ defines the incremental effect of the field on the magnetization vector. That is, the magnetization after a short time interval τ is related to the magnetization at the initial time t by

$$M(t+\tau) = \boldsymbol{F}(\tau)M(t) \tag{14.33}$$

The matrix approach is a concise method that has found utility in a number of applications, including the analysis of excitation (**?**).

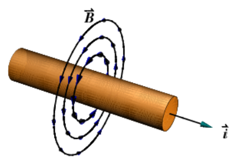
Problems

14.2 Prove Eqn 14.31.

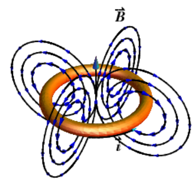
14.6 Coils: Creation and Detection of Magnetic Fields

We now are at an important conceptual juncture. Up to this point, we have been making the distinction between the component in the longitudinal ($\hat{\boldsymbol{z}}$) direction, and the component in the $x-y$ plane (the transverse component). And there is a very important reason for this that we shall see in the next sections: Only the transverse component is what is "seen" (i.e., measured) by our detection equipment, whereas the longitudinal component is not. Later on, we will also see that two components change their magnitude, or *relax*, with different rates. These facts bring up three important issues. The first is: 1) "How do we create transverse magnetization?", since this is what we need to do to measure the signal. The second is: 2) "How do we detect transverse magnetization?". In other words, what is the actual mechanism by which we can make measurements of the magnetization? And thirdly, 3) Is there an efficient way to describe the two components that exist in the three dimensional space of $\boldsymbol{m}(t)$? That is, even though we could describe the magnetization in terms of it three Cartesian components $\boldsymbol{m}(t) = \{m_x(t), m_y(t), m_z(t)\}$, it is clear that there are really just two vectors in which we are interested: the longitudinal component $\boldsymbol{m}_\parallel(t) \equiv \boldsymbol{m}_z(t)$ and the transverse component the $\boldsymbol{m}_\perp(t) \equiv \boldsymbol{m}_{xy}(t)$.

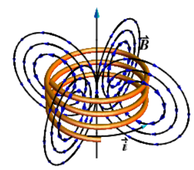
The first two of these are discussed in the next two sections, and will bring us into the area of physics concerned with electric and magnetic fields, or *electromagnetism*. Our discussion on this topic will be brief (for further reading, an excellent introductory text is (**?**)), but it is important to at least grasp the basic ideas since all of MRI is based upon the generation and detection of signals using loops of wires called *coils*.



(a) A straight wire.



(b) A loop of wire.



(c) A solenoid.

Figure 14.3 According to Biot-Savart's Law, a steady current $\vec{i}$ through a wire creates a magnetic field the is perpendicular to the direction of the current. (a) A straight wire. (b) A loop of wire. The arrow (cyan) represent the net field at the center of the loop. (c) A solenoid. The arrow (cyan) represent the net field at the center of the loop. (**Need better depiction of fields here.**)

14.7 Creating a magnetic field: The Biot-Savart Law

The previous sections beg the question "How do you create a static magnetic field?". To answer this, we will see one of the fascinating aspects of physics that actually enters into everyday life. It pertains to the relationship between magnetic fields and electrical currents, this time in a form of *Ampere's Law*, which describes the magnetic field generated by a current running through a closed loop of wire. For a *steady* current I along a very small length of wire $d\boldsymbol{l}$, the differential contribution $d\boldsymbol{B}$ to the magnetic field at some distance r is found from *Biot-Savart's Law*. This is shown in Figure 14.3a. A loop of wire creates the pattern shown in Figure 14.3b. Therefore, by looping a wires around in a helical pattern, called a *solenoid*, and running a current through it, one can create a constant magnetic field pointing along the axis of the helix, as shown in Figure 14.3c.

$$d\boldsymbol{B} = \left(\frac{\mu_o}{4\pi}\right) \frac{I d\boldsymbol{l} \times \hat{\boldsymbol{r}}}{r^2} \tag{14.34}$$

where $\hat{\boldsymbol{r}}$ is a unit vector from the wire along the radius of interest. μ_o is the *magnetic constant* [5]. This is how the main magnetic field is constructed, in fact, as shown in Figure 14.4. As you might imagine, other patterns of magnetic fields can be created with different patterns of wires (**?**).

14.8 Detecting a signal: Faraday's Law of Induction

We have found that if the bulk magnetization is not parallel to the main field $\boldsymbol{B}_o \equiv B_o\hat{\boldsymbol{z}}$, then it will precess about the axis along which $\boldsymbol{B}_o$ is aligned, i.e., the z axis. In order to detect this, we will once again use the relationship between magnetic fields and electrical currents, but in a different form from the last section. We will use the fact that a magnetic field changing with

[5] put its value

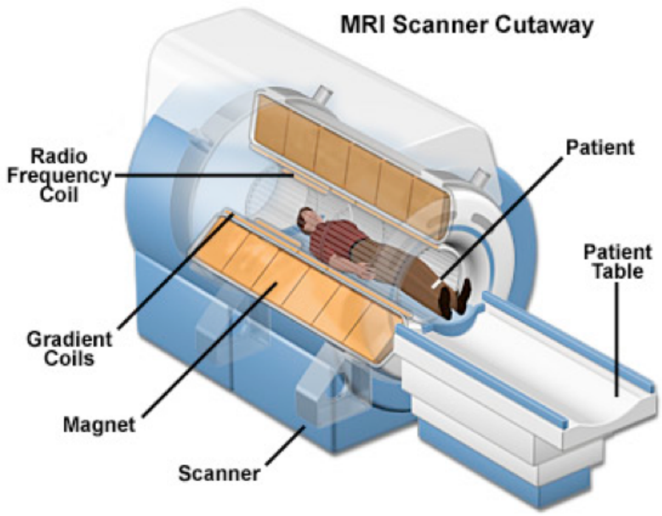


Figure 14.4 The main field of an MRI scanner is created by a solenoid. (diagram from http://www.magnet.fsu.edu/education/tutorials/magnetacademy/mri/)

time generates a current in a surrounding wire. In physics terms, this is stated in the following way: A time rate of change of the *magnetic flux* Φ_B through a closed circuit generates a voltage $V(t)$ according to *Faraday's Law of Induction*:

$$V(t) = -\frac{d\Phi_B}{dt} \tag{14.35}$$

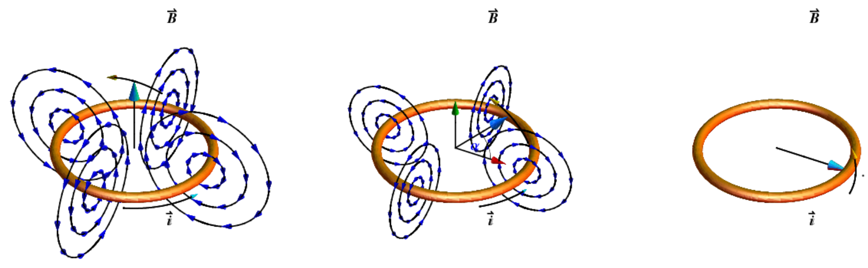
For a magnetic field $\boldsymbol{B}(\boldsymbol{r})$ in the laboratory frame, the flux generated by the precessing bulk magnetization $\boldsymbol{M}(\boldsymbol{r}, t)$ is

$$\Phi_B = \int_\Omega \boldsymbol{B}(\boldsymbol{r}) \cdot \boldsymbol{M}(\boldsymbol{r}, t)\, d\boldsymbol{r} \tag{14.36}$$

and the integral is over the object dimensions Ω.

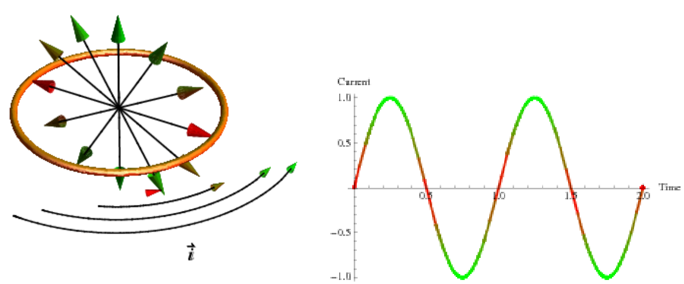
Therefore we see the important fact that we can detect a changing (i.e., precessing) magnetization by the voltage that it induces in a coil, but that this necessitates having the magnetization pointing along a direction other than that which it is naturally aligned: We are detecting the magnetization in the plane perpendicular to $\hat{\boldsymbol{z}}$. That is, it is the *transverse* magnetization $\boldsymbol{m}_\perp$ that is detected, while the longitudinal component $\boldsymbol{m}_\parallel$, does not precess about $\hat{\boldsymbol{z}}$ and thus does not induce a voltage. This is shown in Figure 14.5.

A magnetization that precesses in a plane perpendicular to the plane of the coil thus generates an oscillating field, as shown in Figure 14.7a. The result of this is that a precessing magnetization will induce a signal in a coil that is at any orientation other then in the same plane of the precession, as shown in Eqn 14.7b. We see then that it is necessary to move, or *tip*, the magnetization vector $\boldsymbol{m}$ to a direction other than the direction of the main field if we are to detect a magnetization. This is called *excitation*, and in the next section we see how to accomplish this.



(a) $\alpha = 90°$ (b) $\alpha = 45°$ (c) $\alpha = 0°$

Figure 14.5 Detecting a magnetic field and Faraday's Law. The cyan vector is the total magnetization $\boldsymbol{m}$, moving in a circle and at an instantaneous angle $\alpha°$ relative to the plan of the loop (shown in (b)). The green arrow is the component of $\boldsymbol{m}_\perp$ perpendicular to the coil loop, and the red arrow is the component $\boldsymbol{m}_\parallel$ in the plane of the coil loop. The green component generates a current in the loop, which is what we detect.



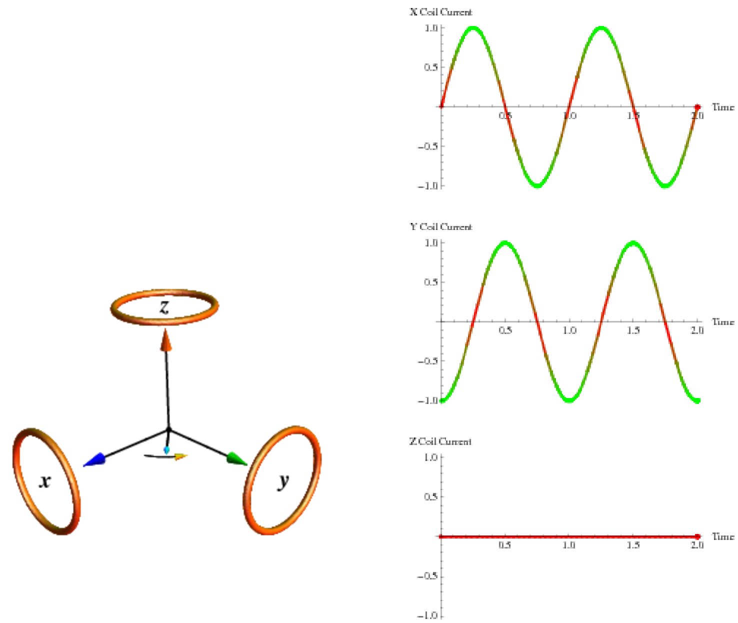
(a) Rotating magnetization. (b) Corresponding signal.

Figure 14.6 Detecting a magnetic field and Faraday's Law. A magnetization vector moving in a plane perpendicular to the loop (a) generates a sinusoidally varying current (b) as it goes from being in the plane of the loop (red) to out of the plane of the loop (green). The magnetization vector and the corresponding signal are colored according to how much of it is in the plane (red) and how much is out of the plane (green).

14.9 Excitation

In the previous section we saw that the component of magnetization that is "seen" by the coils is the transverse component m_{xy}, whereas the magnetization in its equilibrium state in the static magnetic field is aligned along the $\hat{z}$-axis, and thus, if we just let it sit in that field, would be composed of only the longitudinal component m_z. So in order to detect the magnetization we need to find a way to tip the magnetization vector into the transverse plane. This is called *excitation*.

Understanding how to tip over the magnetization is conceptually very simple if you recall our discussion of torque in Section 3.14. Recall our mechanical example of a wrench represented by



(a) A magnetization vector moving in the $x{-}y$ plane and three orthogonal coils perpendicular to each of the three Cartesian axes $\{x, y, z\}$.

(b) The induced voltage in the three coils $\{x, y, z\}$ from top to bottom.

Figure 14.7 A magnetization vector moving in the $x - y$ plane generates signals in the coils perpendicular to x and y, which are 90° out of phase with one another, and no signal in the coils perpendicular to z, which is in the same plane as the rotating magnetization vector.

a vector $\boldsymbol{r}$ (the *moment arm*) initially pointing along the $\hat{z}$ direction. If we pull it down along the $\hat{x}$-axis with force $\boldsymbol{F}$, the torque $\boldsymbol{\tau}$ on the moment arm is

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = rF(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}}) = rF \sin\theta\, \hat{\boldsymbol{y}} \tag{14.37}$$

where $r = |\boldsymbol{r}|$, $F = |\boldsymbol{F}|$ and θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{F}$. If the force is applied perpendicular to the moment arm $\theta = 90^\circ \rightarrow \sin\theta = 1$ and the torque then $\boldsymbol{\tau} = rF\hat{\boldsymbol{y}}$. The moment arm is thus tipped from the longitudinal ($\hat{z}$) direction to the transverse plane, in this case specifically onto the $\hat{y}$-axis. But notice that Eqn 14.2 is just such an equation, with the magnetic field $\boldsymbol{B}$ applying a torque $\boldsymbol{\tau} = d\boldsymbol{M}/dt$ on the magnetization vector $\boldsymbol{M}$. We can immediately conclude, by analogy, that if we apply a magnetic field $\boldsymbol{B}_1(t)$ perpendicular to the magnetization vector (say, along the $\hat{x}$ direction) we can rotate it from the longitudinal direction to the transverse plane, onto the $\hat{y}$ axis. But there is a complication - the magnetization vector is precessing at the Larmor frequency ω_0 so applying the field $\boldsymbol{B}_1$ along the x-axis of the magnet (the laboratory frame) would mean that the angle θ in Eqn 14.37 was constantly changing. It would be like trying to tighten the bolt on a rotating piece of equipment. But the solution to this is easy: If you want to tighten a bolt on a horse on a moving Merry-Go-Round, you don't do it from the ground outside it, you hop onto the Merry-Go-Round. Once you are in the rotating reference frame of the horse, the problem becomes easy. So that's the answer - we need to apply the magnetic field $\boldsymbol{B}_1$ in the Larmor rotating frame.

Technically, it is not a problem to apply a magnetic field that rotates at some angular frequency Ω_1. The field we want to create rotates in both the $\hat{z}$ (the direction of the rotating frame) and the $\hat{x}$ (the direction we want to tip the magnetization onto the $\hat{y}$ axis). Using Eqn 14.4,

$$\boldsymbol{\Omega}_1 = \gamma B_{1,x}\boldsymbol{A}_x + \gamma B_{1,z}\boldsymbol{A}_z = \begin{pmatrix} 0 & -\omega_{1,z} & 0 \\ \omega_{1,z} & 0 & -\omega_{1,x} \\ 0 & \omega_{1,x} & 0 \end{pmatrix} \tag{14.38}$$

From Eqn 14.22 the Bloch equations in the Larmor rotating frame are

$$\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} = (\boldsymbol{\Omega}_1 - \boldsymbol{\Omega}_o)\boldsymbol{M} + \boldsymbol{R}\boldsymbol{M} \tag{14.39}$$

where, from Eqn 14.20 and Eqn 14.38

$$\boldsymbol{\Omega}_1 - \boldsymbol{\Omega}_o = \begin{pmatrix} 0 & \omega_0 - \omega_{1,z} & 0 \\ -(\omega_0 - \omega_{1,z}) & 0 & -\omega_{1,x} \\ 0 & \omega_{1,x} & 0 \end{pmatrix} \tag{14.40}$$

From this it is clear that if we set the angular frequency in $\hat{z}$ to the Larmor frequency, $\omega_{1,z} = \omega_o$ and set the angular frequency along $\hat{x}$ to $\omega_{1,x} = \gamma B_1 \hat{x}$ where B_1 is a constant, then

$$\boldsymbol{\Omega}_{\text{rf}} \equiv \boldsymbol{\Omega}_1 - \boldsymbol{\Omega}_o = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\gamma B_1 \\ 0 & \gamma B_1 & 0 \end{pmatrix} \tag{14.41}$$

Because the applied field $\boldsymbol{B}_1(t)$ rotates about $\hat{z}$ at the Larmor frequency, which for MRI is in the *radio-frequency* or *RF* range of frequencies ($\approx 30Khz - 300Ghz$), this pulse is typically referred to as the *RF-pulse* and the frequency designated with a subscript "rf", as we've done in Eqn 14.41. Thus, spins are tipped when one applies a field in the lab frame that rotates at the Larmor frequency. This condition is called *resonance* and so this type of excitation is called *on-resonance excitation*. Because this involves the magnetic moment of the nucleus (protons), this type of resonance is called *nuclear magnetic resonance*, or *NMR*.

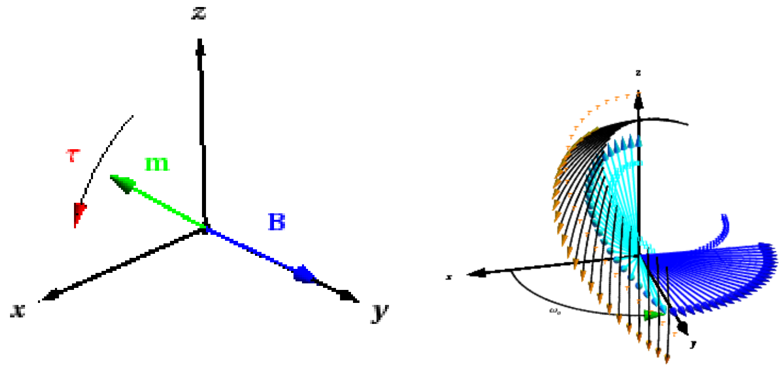
The tipping of the magnetization in the two frames is illustrated in Figure 14.8. The tipping of the magnetization occurs during the application of the RF pulse $\boldsymbol{B}_1(t)$. The degree to which it is tipped is characterized by the size of the angle α, the *flip angle*, between the magnetization vector and the $\hat{z}$ axis. The flip angle depends on the length of time τ the field is kept on, and the field strength, $B_1 = |\boldsymbol{B}_1|$:

$$\alpha = \int_0^\tau \omega_1(t)\, dt = \gamma \int_0^\tau B_1(t)\, dt \tag{14.42}$$

For a pulse that is constant in the rotating frame

$$\alpha = \gamma B_1 \int_0^\tau dt = \gamma B_1 \tau \tag{14.43}$$

Since the flip angle depends upon the time integral and the magnitude of the field, the value of the flip angle can be altered either by changing the strength of the field $B_1(t)$ or its duration. For example, for the special case of a constant B_1 of fixed amplitude, doubling the time pulse duration τ produces a flip angle twice as big. An example of a 90° pulse is shown in Figure 14.9.



(a) The torque $\boldsymbol{\tau}$ on $\boldsymbol{m}$ produced by $\boldsymbol{B}$. (b) Trajectory of $\boldsymbol{m}$ under the influence of $\boldsymbol{B}$.

Figure 14.8 To keep the B-field perpendicular to the precessing magnetization, it must rotate at the same frequency. In (a) is shown the torque $\boldsymbol{\tau}$ on the magnetization $\boldsymbol{m}$ produced by the magnetic field $\boldsymbol{B}$. In (b) is shown the trajectory of the magnetization under the influence of $\boldsymbol{B}$. At each time step the magnetization $\boldsymbol{m}$ has precessed (i.e., rotated) relative to the previous time and so the magnetic field $\boldsymbol{B}_1$ necessary to tip it to the transverse plane must also rotate in step. Magnetization is in cyan, applied field is in blue, and torque is in orange.

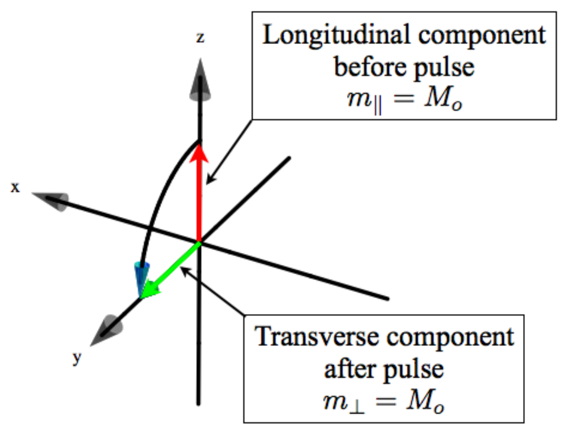
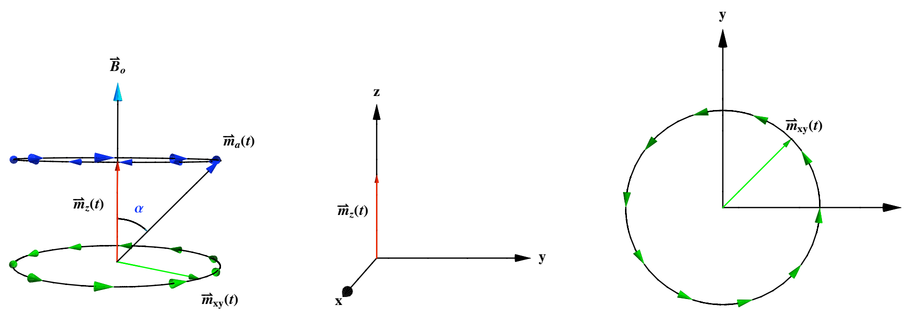


Figure 14.9 A 90° pulse in the rotating frame.

In the laboratory frame, an RF pulse arbitrary of flip angle α creates a transverse component of the magnetization $\boldsymbol{m}_{xy}$, its projection onto the transverse plane, and a longitudinal component $\boldsymbol{m}_z$, its projection onto the longitudinal axis. This is illustrated in Figure 14.10. It is worth returning now briefly to that common area of confusion, brought up way back in Section 24.1, concerning how one can possibly have a flip angle of arbitrary degree when spins can have only two orientations, "up" and "down". At this point it should be clear that there really is no problem



(a) Precession of $\boldsymbol{m}$ about $\boldsymbol{B}_o$ (b) $\boldsymbol{m}_z$ (c) $\boldsymbol{m}_{xy}$

Figure 14.10 Motion of the magnetization that is initially at an angle α with the main field $\boldsymbol{B}_o = B_o\hat{\boldsymbol{z}}$. In (a) is shown precession of the magnetization $\boldsymbol{m}$ (blue) about the static magnetic field $\boldsymbol{B}_o$ (cyan).(**This is in the reverse direction as TTL!**). In (b) is shown longitudinal component $\boldsymbol{m}_z$ of the magnetization. In (c) is shown the transverse component $\boldsymbol{m}_{xy}$ of the magnetization.

here at all. We long ago (Chapter 13) demonstrated that the collective behavior of the spins system was sufficiently described by a classical magnetization vector whose motion is described by the Bloch Equations and whose orientation can be altered by solving these equations in the presence of external fields other than the main static field. The magnetization is simply a classical vector that can be driven through any flip angle and take on *any* orientation.

14.10 The Magnetization Trajectory

A useful device for investigating the influence of an RF pulse on the magnetization is to plot the path traced out by $\boldsymbol{M}(t)$, called the *trajectory*, as a function of time in the presence of the applied field $\boldsymbol{B}_1(t)$. This can be thought of as plotting the tip of the magnetization vector. A useful device to visualize the trajectory is to plot it on the sphere whose radius is the length of the (unrelaxed) magnetization vector. This is called the *Bloch sphere*. The magnetization can be plotted in both the rotating frame and the laboratory frame, as is done Figure 14.11 for the case of $\alpha = 180°$, where the magnetization vector is rotated from $+\hat{z}$ to $-\hat{z}$. This is also called an *inversion pulse*. To determine $\boldsymbol{M}(t)$, the Bloch equation needs to be solved. This is made much easier if it is possible to apply the fields necessary for our manipulation of the magnetization much more rapidly than the relaxation process. That is, if τ is the time need to apply an external excitation field, then $\tau \ll T_2$ (and thus also T_1 since $T_2 < T_1$) and the Bloch equation becomes

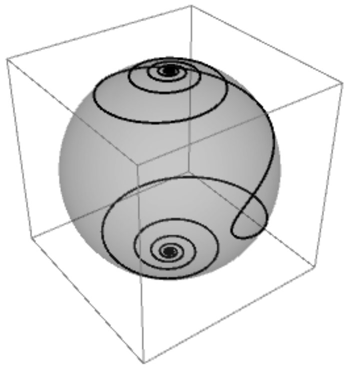
$$\frac{d\boldsymbol{M}}{dt} + \boldsymbol{\Omega}\boldsymbol{M} = 0 \tag{14.44}$$

which is easy to solve:

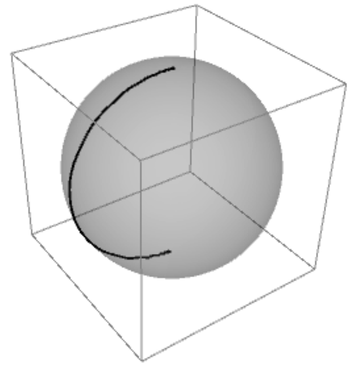
$$\boldsymbol{M}(t) = M(0)e^{-\boldsymbol{\Omega}t} \tag{14.45}$$

[6]

[6] Make connection here with infinitesimal rotations and rotation matrices!!



(a) Path of the magnetization in the laboratory frame.



(b) Path of the magnetization in the rotating frame.

Figure 14.11 The path of the magnetization for a 180° pulse in the laboratory and the rotating frame.

The general solution to the Bloch Equation in the rotation frame can then be written[7]

$$\boldsymbol{m}_{rot}(t) = \boldsymbol{R}(\alpha)\boldsymbol{m}(0) \tag{14.46}$$

where α is given by Eqn 14.42. The rotation can be applied about any arbitrary axis and from our discussion of rotation matrices in Section **??** we know that we can rotate a vector about any angle by a combination of rotations about the three Cartesian coordinate $\{x', y', z'\}$ (here, the rotating Cartesian coordinate of the rotating frame). These rotations are just our familiar rotation matrices given in Eqn 6.10. In practice, the majority of RF pulses are applied along one of the three orthogonal axes $\{x', y', z'\}$, which nicely simplifies the description of a series of pulses in practical applications. For example, an RF pulse of angle α_x along x, followed by a pulse of angle α_y along y, and then one of angle α_z along z can be calculated simply as

$$\boldsymbol{m}_{rot}(t) = \boldsymbol{R}_z(\alpha_z)\boldsymbol{R}_y(\alpha_y)\boldsymbol{R}_x(\alpha_x)\boldsymbol{m}(0) \tag{14.47}$$

From the discussion in Section **??** we know that the product of the three rotation matrices is a single rotation. A pulse can also be applied with a phase angle ϕ which is easily incorporated into Eqn 14.47 by an addition rotation by ϕ about $\hat{z}$.

Plotting the trajectory in the lab frame (e.g., Figure 14.11a) can be accomplished by solving the Bloch equations in the lab frame. The use of rotation matrices makes moving between the lab frame and the rotating frame very easy. For a pulse of duration τ, the rotating frame magnetization can be expressed in terms of the lab frame as

$$\boldsymbol{m}_{rot}(t) = \boldsymbol{R}(\omega_o\tau)\boldsymbol{m}_{lab}(t) \tag{14.48}$$

But now we can use the fact that rotation matrices are orthogonal matrices (Section 5.16), i.e., $\boldsymbol{R}^{-1}\boldsymbol{R} = \boldsymbol{I}$ means that we can multiply Eqn 14.48 on the left by $\boldsymbol{R}^{-1}$ to get the transformation

[7] The next step we haven't shown - the rotation matrix solution for the general Bloch equation - i.e., for a general B1 pulse!

from rotating frame to lab frame:

$$\boldsymbol{m}_{lab}(t) = \boldsymbol{R}(\omega_o \tau)^{-1} \boldsymbol{m}_{rot}(t) \tag{14.49}$$

The use of rotation matrices to determine the trajectory of the magnetization under the influences of multiple pulses is exceedingly useful, and made more so by the ability of modern computation mathematics tools (e.g., Mathematica (**?**)) to symbolically accept such input and to efficiently calculate it numerically.

14.11 Off-resonance excitation

In the previous section we saw that excitation can be achieved by applying a magnetic field $\boldsymbol{B}_1(t)$ with a component perpendicular to the $\hat{z}$-axis and rotating about the $\hat{z}$-axis at the Larmor frequency. This was called on-resonance excitation. But what happens if a spin is not in a field B_o but one that is slightly different, $B_o + \delta B$? This situation is not uncommon: there are a variety of conditions that will create a variation ΔB in the local magnetic field and result in spins precessing at a rate $\omega_o + \Delta\omega = \gamma(B_0 + \Delta B)$. In this case, there is a mismatch between the precessional frequency of the spins $\omega = \omega_o + \Delta\omega$ and the angular frequency ω_o of the applied field $\boldsymbol{B}_1(t)$. Excitation in this case is called *off-resonance excitation*.

Again using Eqn 14.4, with Eqn 14.38 as before, we can define

$$\boldsymbol{\Delta\Omega} = \gamma \Delta B_{0,z} \boldsymbol{A}_z = \begin{pmatrix} 0 & -\Delta\omega_{0,z} & 0 \\ \Delta\omega_{0,z} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14.50}$$

Assuming now that the $\hat{z}$ component of Ω_1 is at the Larmor frequency, from Eqn 14.22 the Bloch equations in the Larmor rotating frame are

$$\left(\frac{d\boldsymbol{M}}{dt}\right)_{rot} = (\boldsymbol{\Omega}_1 + \Delta\boldsymbol{\Omega}_o)\boldsymbol{M} + \boldsymbol{R}\boldsymbol{M} \tag{14.51}$$

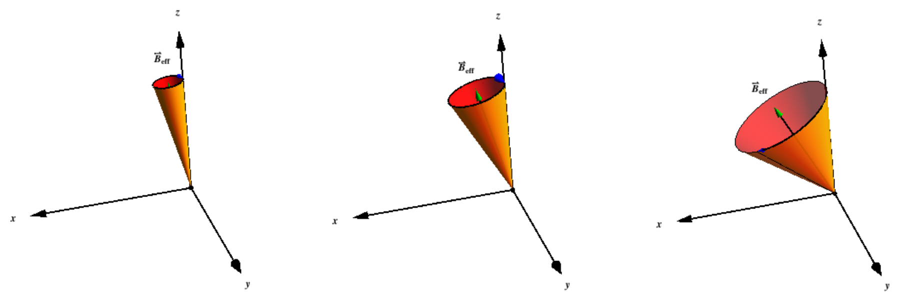
where, from Eqn 14.20 and Eqn 14.38

$$\boldsymbol{\Omega}_1 + \boldsymbol{\Delta\Omega}_o = \begin{pmatrix} 0 & -\Delta\omega_0 & 0 \\ \Delta\omega_0 & 0 & -\omega_{1,x} \\ 0 & \omega_{1,x} & 0 \end{pmatrix} \tag{14.52}$$

gain neglecting relaxation ($\boldsymbol{R} = \boldsymbol{0}$), on the grounds that the relaxation times are much longer than the pulse duration ($\tau \ll T_1, T_2$).

14.12 Off-resonance excitation (old)

In the previous section we found that for a group of spins at the frequency ω_o of the main field, the resonance frequency was just $\omega_o = \gamma B_o$. So that by setting $\boldsymbol{\omega} = -\omega_o \hat{\boldsymbol{z}}'$, the resonance condition is achieved. But what happens if a spin is not in a field B_o but one that is slightly different, $B_o + \delta B$? In fact, this situation is not uncommon: there are a variety of conditions



(a) Off-resonance excitation $\omega_1/\Delta\omega_o = 1/2$. (b) Off-resonance excitation $\omega_1/\Delta\omega_o = 1/4$. (c) Off-resonance excitation $\omega_1/\Delta\omega_o = 1/8$.

Figure 14.12 Off resonance excitation.

that will create a variation in the local magnetic field. A spin in this field will precess at a rate $\omega_o + \Delta\omega$. So in the rotating frame, what does this look like? From Eqn **??** with $\boldsymbol{\omega} = \omega_0 \hat{\boldsymbol{z}}'$

$$\boldsymbol{B}_e(t) = \frac{\Delta\omega_o}{\gamma}\hat{\boldsymbol{z}}' + B_1(t)\hat{\boldsymbol{y}}' \tag{14.53}$$

This can be solved for a constant B_1 applied for a time τ [8]

$$M_{x'}(t) = M_z^o \sin\theta\cos\theta\,(1 - \cos(\omega_e t)) \tag{14.54a}$$
$$M_{y'}(t) = M_z^o \sin\theta\sin(\omega_e t) \tag{14.54b}$$
$$M_{z'}(t) = M_z^o \left(\cos^2\theta + \sin^2\theta\cos(\omega_e t)\right) \tag{14.54c}$$

The effective field $B_e = \omega_e/\gamma$ where

$$\omega_e = \sqrt{\Delta\omega_o^2 + \omega_1^2} \tag{14.55}$$

and is at an angle θ between the z' axis and the axis along which B_1 is oriented (here x')

$$\theta = tan^{-1}\left(\frac{\omega_1}{\Delta\omega_o}\right) \tag{14.56}$$

The situation is shown in Figure 14.12. At then end of the pulse, i.e., at $t = \tau$, we replace $\omega_o t$ with α, the flip angle, in Eqn 14.54 to get the final magnetization vector. From this we can calculate the final magnitude and phase of the magnitization:

$$M_\perp = M_z^o \sin\theta\sqrt{\sin^2\alpha + (1-\cos\alpha)^2\cos^2\theta} \tag{14.57a}$$
$$\varphi = \frac{M_x}{M_y} = \frac{\alpha}{2}\frac{\Delta\omega_o}{\omega_e} \tag{14.57b}$$

The magnitude decreases as the frequency offset increase [9] whereas the phase shift increases with the frequency shift $\Delta\omega_o$.

[8] Gottfried gives a cleaner solution than Liang 3.107!
[9] Show plot?

Problems

14.3 Prove Eqn **??**.

Problems

14.4 Prove Eqn 14.57

Problems

14.5 Prove Eqn 14.57

14.13 The NMR signal

Once we have created a transverse magnetization by excitation, this precessing magnetization can be detected by the voltage that it induces in the coil. The form of this signal depends on the specific details of the detection equipment (a good discussion of processing is given in (**?**)) but for most MR scanners with a receiver that is homogeneous over the object, the detected signal can be processed in such a way that we detect a signal

$$s(t) = \int_\Omega \boldsymbol{m}_\perp(\boldsymbol{r}, t) e^{-i\omega(\boldsymbol{r},t)}\, d\boldsymbol{r} \tag{14.58}$$

where $\boldsymbol{r}$ is the vector in 3-dimension Cartesian space, i.e., $\boldsymbol{r} = \{x, y, z\}$ and $\omega(\boldsymbol{r}, t)$ is the frequency *in the rotating frame*. That is, the *difference* between the frequency in the lab frame and the resonance frequency:

$$\omega(\boldsymbol{r}, t) = \gamma \int_0^t B(\boldsymbol{r}, \tau)\, d\tau \tag{14.59}$$

Therefore we only need concern ourselves with *deviations* of the field from B_o and of frequencies from ω_o.[10]

Notice the very important fact expressed by Eqn 14.58:

The signal is the Fourier Transform of the transverse magnetization

(see Section **??** if you have forgotten the Fourier Transform.) This fact is central to so many aspects of MRI that its importance cannot be over-emphasized. In addition to providing the method of reconstructing the images from the data, it also motivates the methods of data acquisition and explains the structure of artifacts, among many other things.

In the last section we came upon two important characteristics of the signal that need to be emphasized here. First, the signal is generated by the motion of the magnetization in the $x-y$ plane, the *transverse component* $\boldsymbol{m}_\perp$, but not the component along the z-axis, the *longitudinal component* $\boldsymbol{m}_\parallel$. This means that the natural way to described the magnetization is *not* in terms of its three Cartesian components $\boldsymbol{m} = \{\boldsymbol{m}_x, \boldsymbol{m}_y, \boldsymbol{m}_z\}$ but in terms of its two components $\boldsymbol{m} = \{\boldsymbol{m}_\parallel, \boldsymbol{m}_\perp\}$. Secondly, the component $\boldsymbol{m}_\perp$ is just a vector in the $x-y$ plane rotating about the z-axis, and its x and y components are 90° out of phase. This suggests that a natural representation of the transverse component is in terms of the *complex numbers* that we discussed in Chapter 4. Recall from that discussion that this description is convenient because instead of keeping track of two parameters (x and y), we need only keep track of one, the phase angle,

[10] add argument from class slides.

which then tells us what those two parameters must be. That is, in Figure **??**, $\boldsymbol{m}_{xy} = \boldsymbol{m}_x + i\boldsymbol{m}_y$ where the angle between $\boldsymbol{m}_{xy}$ and the x-axis is $\phi = \tan^{-1}(y/x)$ (see Eqn **??**). This is also the reason for the complex dependence on ω (i.e., $e^{-i\omega}$) in Eqn 14.58.

Now, notice that the signal depends on the transverse magnetization, which is subject to T_2 relaxation according to Eqn 13.36. Substituting this into Eqn 14.58, the signal equation becomes

$$s(t) = \int_{\Omega} \boldsymbol{m}_{\perp}(\boldsymbol{r}, 0) e^{-t/T_2} e^{-i\omega(\boldsymbol{r},t)} \, d\boldsymbol{r} \tag{14.60}$$

In summary, the motion of the magnetization and which of its component are detectable suggests that a natural representation of the magnetization is in terms of two components, the longitudinal component $\boldsymbol{m}_{\parallel}$ which is a real vector, and the transverse component $\boldsymbol{m}_{\perp} = \boldsymbol{m}_x + i\boldsymbol{m}_y$ that is a complex number. If it looks strange to have the magnetization expressed in terms of two different types of vectors (one real, one complex), well, you're right.

Eqn 14.60 says that the signal detected in the RF coils from the transverse magnetization of the freely precessing spin (i.e., not in the presence of any excitation pulses, just the main field) decays as the transverse magnetization is reduced by T_2 effects, and so is called the *free induction decay* or *FID* and is shown in Figure 14.13 for a single isochromat. To summarize, immediately following the termination of the excitation pulse:

1. Spins precess in the main field **Free** from excitation pulses
2. The precessing spin generates current in the RF coils by Faraday's Law of **Induction**
3. This signal diminishes exponentially due to **Decay** of the transverse component of magnetization.

14.14 Isochromats and T_2^* relaxation

The main field precessional frequency is typically called the *resonance frequency*, since that is the frequency at which an excitation pulse must be applied to resonate the spins. Ideally, all spins in a the main magnetic field $\boldsymbol{B}_o = B_o\hat{z}$ would have the same resonance frequency, given by the Larmor equations:

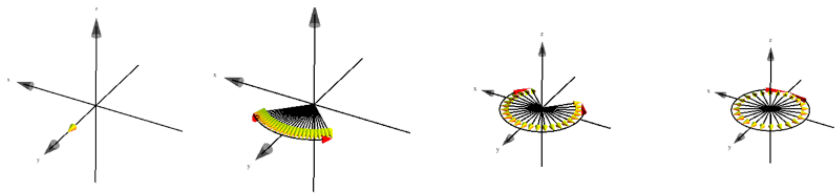
$$\omega_o = \gamma B_o \tag{14.61}$$

In practice, however, spins possess a range of resonance frequencies. There are two primary reasons for this. First, the main field can be inhomogeneous. If this field deviates from the ideal value of B_o by an amount ΔB_o, then the local precessional frequency will deviate by an amount $\Delta\omega_o = \gamma\Delta B_o$. And thus the actual resonance frequency will be

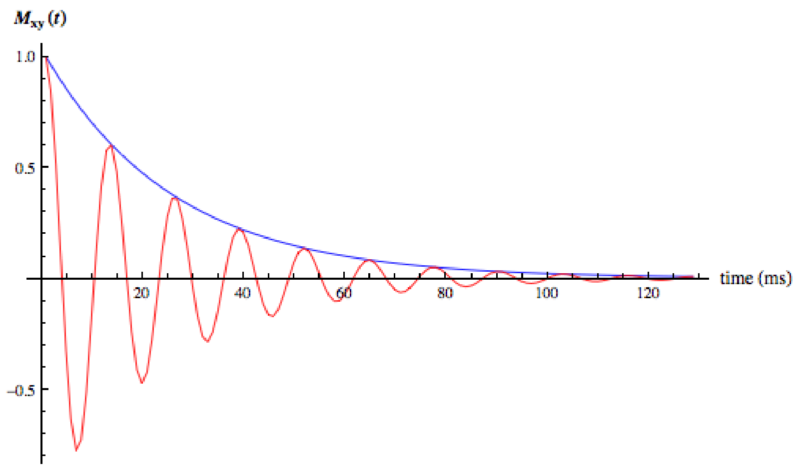
$$\omega_{resonance} = \omega_o + \Delta\omega_o = \gamma(B_o + \Delta B_o) \tag{14.62}$$

The second reason for variations in resonance frequencies is that spins in an actual tissues are part of complex molecular environments wherein the orbital electrons of the molecules produce small magnetic fields that tend to be in the opposite direction of the applied fields[11], and thus the nucleus sees a field that is slightly less than the applied field. It is usually said that the electrons "shield" the nucleus from the external field. This effect is typically characterized by

[11] explain more?



(a) Transverse magnetization of spins contributing to the FID.



(b) Free induction decays for a single isochromat. Blue line is on-resonance, red line is off-resonance

Figure 14.13 Decay of the transverse magnetization according to (Eqn 13.36) due to the dephasing depicted in (a). This is called the *free induction decay.* For spins on-resonance, the curve is a simple decay (blue) but for spin off-resonance the signal oscillates and decays. The envelope of the signal is the on-resonance decay curve.

introducing the *shielding constant* δ that is the strength of the offset field as a fraction of the main field so the field the nucleus experience is expressed as

$$B_{o,true} = B_o - \delta B_o = B_o(1 - \delta) \tag{14.63}$$

and thus the resonance frequency is

$$\omega_{o,true} = (1 - \delta)\omega_o \tag{14.64}$$

Since there are variations in the resonance frequencies, a useful concept is that of a group of spins with the *same* resonance frequency, which is called an *isochromat*.

In general, there can be many different frequencies within a small volume of tissue, so the only practical way to characterize them is by their distribution. In this case Eqn 14.60 can be rewritten in terms of the frequencies as

$$s(t) = \int_\Omega \boldsymbol{m}_\perp(\omega, 0) e^{-t/T_2} e^{-i\omega(\boldsymbol{r},t)}\, d\omega \tag{14.65}$$

A distribution of frequencies is called the *spectrum* of frequencies. The spectrum of frequencies is typically modeled as a Lorentzian distribution

$$f(\omega) = \boldsymbol{m}_{z,n}^{eq} \frac{(\gamma \Delta B_o)^2}{(\gamma \Delta B_o)^2 + (\omega - \omega_o)^2} \tag{14.66}$$

where $\boldsymbol{m}_{z,n}^{eq}$ is the equilibrium bulk magnetization at frequency n. Substituting Eqn 14.66 into Eqn 14.65 gives

$$\begin{aligned} s(t) &= \int_\Omega \boldsymbol{m}_{z,n}^{eq} \left[\frac{(\gamma \Delta B_o)^2}{(\gamma \Delta B_o)^2 + (\omega - \omega_o)^2} \right] e^{-t/T_2} e^{-i\omega(\boldsymbol{r},t)}\, d\omega \\ &= \pi(\gamma \Delta B_o) \boldsymbol{m}_{z,n}^{eq} e^{-t/T_2^*} e^{-i\omega_o t} \end{aligned} \tag{14.67}$$

where [12]

$$\frac{1}{T_2^*} = \frac{1}{T_2} + \gamma \Delta B_o \tag{14.68}$$

The decay of the transverse magnetization according to Eqn 13.36 and characterized by T_2 is an empirical model for the signal loss due to molecular processes, and is sometimes called the *natural* decay process. The new decay constant T_2^* (pronounced "T2 star") is shorter than (or equal to) T_2, which is always shorter than (or equal to) T_1. That is,

$$T_2^* \leq T_2 \leq T_1 \tag{14.69}$$

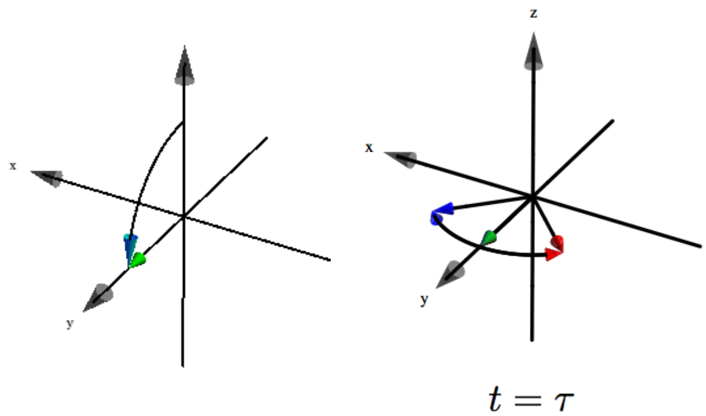
The process of T_2^* relaxation is shown in Figure 14.14 In subsequent chapters we will see that T_2^* decay can be quite a problem in DTI, as it causes signal loss in regions of field inhomogeneities. However, it can also be used to advantage in some applications, such as FMRI, where the promotion of signal loss due to brain activity induced field inhomogeneities is actually desired to enhance functional contrast.

14.15 Excitation (RF) Echoes

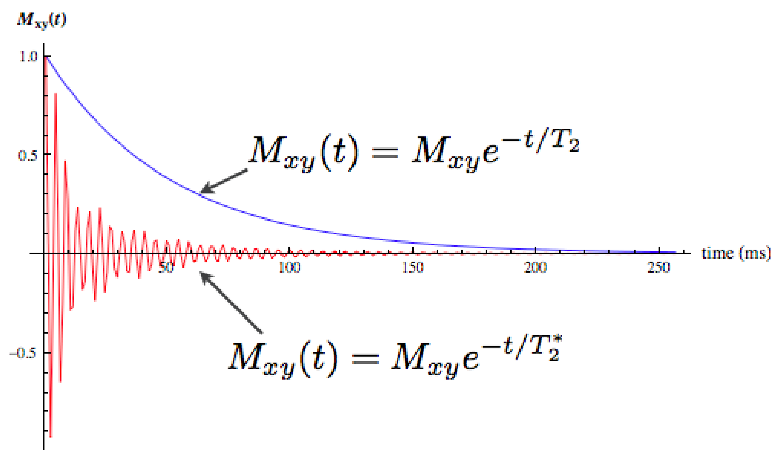
In the previous section we augmented our description of the motion of the magnetization with the relaxation that results from the aggregated behavior of a large number of spins. In particular we found that following excitation, by which we mean tipping the initial longitudinal magnetization into the transverse plane, the transverse magnetization decayed away while the longitudinal magnetization grew back. The effects are summarized in Figure 13.9. However, we also found that while the transverse magnetization decayed away with a "natural" time constant T_2 (Figure 14.13b), in practice it often decays away more quickly, with a shorter time constant T_2^* that results from main field inhomogeneities (Figure 14.13a).

In this section we consider what happens when we add additional excitation, or RF, pulses after the initial 90° excitation pulse. It turns out that while the natural T_2 decay is an intrinsic property of the tissue, the effects that cause T_2^* decay *can* be corrected for. With the addition of just a single well-constructed RF pulse, we will see one of the most remarkable properties of NMR experiment: the ability to recover "lost" signal by the process known as *excitation echoes* or *rf echoes*.

[12] next eqn missing a 1/2? Check!



(a) Excitation pulse 90°_y. (b) T_2^* dephasing.



(c) The FID with T_2^* included. T_2 is irreversible intrinsic transverse decay. T_2^* include reversible decay due to field inhomogeneities.

Figure 14.14 T_2^* dephasing.

14.15.1 The Spin Echo

Consider now a collection of spins at different resonance frequencies, i.e., several isochromats, that are initially aligned along the main field $\hat{z}$, then excited with a 90°_y excitation pulse so that they are all aligned along the x-axis, as shown in Figure **??**. After the excitation pulse is turned off, the spins are allowed to undergo free precession for a time τ. During this time, the spin precess according to the local field they are in. Spins in different fields precess through different angles φ in the transverse plane during a time τ: a spin at a higher field precesses faster, and thus goes through a larger angle than a spin at a smaller field for the same time interval τ. Plotting all spin vectors together (although they inhabit different physical locations) the increasing phase accrual and discrepancy from one another is shown in Figure 14.15. At time τ, a pulse that rotates the spins 180° about the x-axis is applied, as shown in Figure **??**. A spin that had accrued a phase φ is now made to have a phase of $-\varphi$. It is still precessing in the same direction and at the same rate, so after another time τ later its phase exactly opposite that accrued during the first interval, and so cancels exactly. This does not depend on the value of φ.

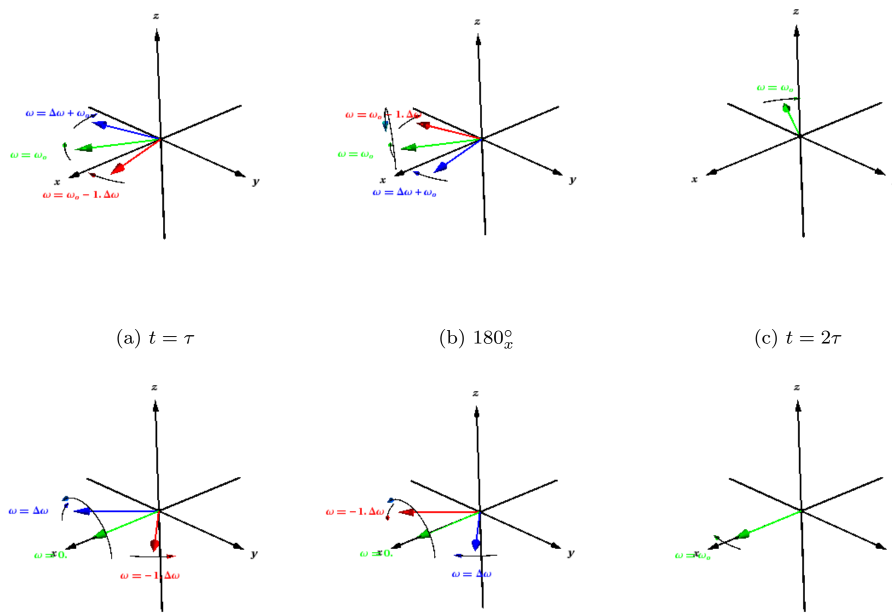
The spin echo is shown in both the laboratory frame and the lab from in Eqn 14.15. It is shown in these two frames because simple diagrams explaining the spin echo unnecessarily lead to another one of the major sources of confusion for beginning students. The steps of the spin echo are as follows.

1. First an excitation pulse tips the magnetization into the transverse plane, as shown in Figure **??** where a 90°_y is used for excitation.
    1. Consider the phases at $t = \tau$ in the lab frame as shown in Figure 14.15d . The red spin sees a slightly larger field $B_o + \Delta B_o$ and precesses faster than the blue spin, which sees a slightly lower field $B_o - \Delta B_o$.
    2. Consider the phases at $t = \tau$ in the rotating frame as shown in Figure 14.15d. Red spin's phase $\omega_o + \Delta\omega_o$ is slightly larger than ω_o and so advances relative to on-resonance spins, whereas the blue spin's phase $\omega_o - \Delta\omega_o$ is slight smaller than ω_o and so retreats relative to on-resonance spins.
2. After a time interval τ, a 180°_x is applied.
    1. Refocussing pulse 180°_x in the lab frame (Figure 14.15b)
    2. Refocussing pulse 180°_x in the rotating frame (Figure 14.15e).
3. We wait a time τ after the 180°_x.
    1. Phases at $t = \tau$ in the lab frame immediately *following* the 180° pulse (assumed to take no time to apply) shown in (Figure 14.15b) . Spins are always precessing in the same direction. But the spins' phases go from φ to $-\varphi$ so the larger red spin phase means that it is flipped to a larger negative phase, and is thus farther away from the x-axis than the blue spin. The red spin has farther to go, but it is precessing faster.
    2. Phases at $t = \tau$ in the rotating frame immediately *following* the 180° pulse (assumed to take no time to apply) shown in (Figure 14.15e). The spins are always precessing in the same direction but in the rotating frame the blue and red arrow refer to the *difference* in phase from $\omega_o t$. Therefore the red spin appears to advance while the blue spin recedes.
4. At a time 2τ after the excitation, a spin echo occurs.
    1. Phases at $t = 2\tau$ in the lab frame (Figure 14.15c). Both spins come into alignment with the on-resonance rotating vector.
    2. Phases at $t = 2\tau$ in the rotating frame (Figure 14.15f). Phases at $t = 2\tau$ in the rotating frame. Both spins come into alignment with x'-axis.

Therefore, at the end of the sequence of pulses,

$$90^\circ - \tau - 180^\circ - \tau - echo \tag{14.70}$$

at the time 2τ, all of the spins come into phase, or *refocus*. For this reason, the 180°_x pulse is called a *refocussing pulse*. The spins coming into phase is called a *spin coherence* because it results in the spins' signals adding coherently, forming a net signal that is called a *spin echo*. This effect was discovered by Erwin Hahn (**?**), one of the early pioneers of NMR, and so is also called a *Hahn echo*. If the initial magnetization was M_o, then at time $t = 2\tau$ the signal has decayed only



(a) $t = \tau$ (b) 180°_x (c) $t = 2\tau$

(d) $t = \tau$ (e) 180°_x (f) $t = 2\tau$

Figure 14.15 The spin echo seen in both the lab (top) and rotating (bottom) frames following the excitation pulse Figure 14.14a. In the rotating frame the refocussed spins appear to be going in the reverse direction, but this is only *relative* to the Larmor frequencies. *Spins always precess in the same direction.* In (a) is shown th phases at $t = \tau$ in the lab frame. In (b) is shown the efocussing pulse 180°_x in the lab frame. In (c) is shown phases at $t = 2\tau$ in the lab frame. Both spins come into alignment with the on-resonance rotating vector. In (d) is shown the phases at $t = \tau$ in the rotating frame. In (e) is shown the refocussing pulse 180°_x in the rotating frame. In (f) is shown the phases at $t = 2\tau$ in the rotating frame. Both spins come into alignment with x'-axis.

by T_2 rather than T_2^*, and thus has a magnitude of $M_o e^{-2\tau/T_2}$ rather than $M_o e^{-2\tau/T_2^*}$. What is remarkable about this effect is that after a time $t \gg T_2^*$, *there is no obverable signal.* And yet, encoded in the spin phases is an order that can be recovered, from which a signal can be generated.

14.15.2 Multiple epin echoes

At a time τ after the application of the 180° pulse (that is, a time 2τ after the excitation) the spins that are precessing at different rates come back into phase. But what happens after that? Well, the spins are still precessing at different rates, so they continue to get out of phase. In fact, the situation at time $t = 2\tau$ look identical to the initial situation at time $t = 0$ immediately following the excitation (90°) pulse, except for the T_2 decay by an amount $e^{-2\tau/T_2}$ from the initial magnetization. Therefore if we apply another 180°_y pulse after another interval τ (a total

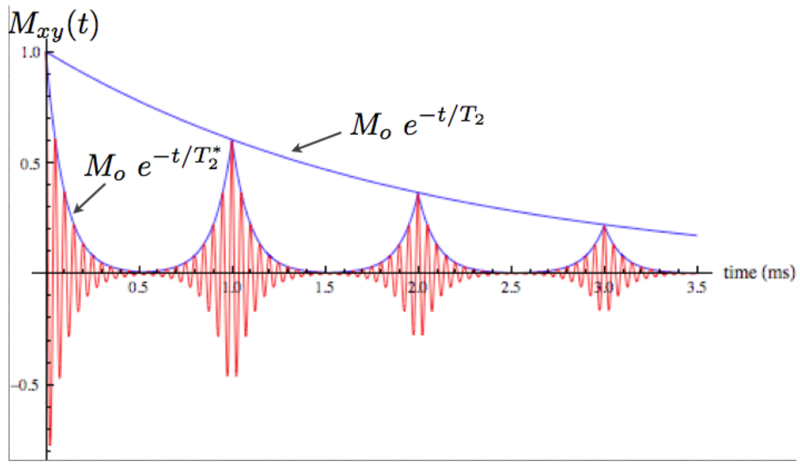


Figure 14.16 Multiple spin echoes. The T_2^* is refocussed by the spin echoes so that the envelope of the echos decays according to $T_2 > T_2^*$.

time 3τ after excitation) then we will get another spin echo after another interval 2τ (a total time 4τ after excitation). By doing this, we can create a *spin echo train* where each of the echoes is produced by the refocusing of the T_2^* effects. The echoes are still subject to the inevitable T_2 effects and so the echo train amplitudes still decay with amplitudes $M_o e^{-2n\tau/T_2}$ where n is the echo number. This is shown in Eqn 14.16. A moments thought will convince you that the intervals don't all have to be the same length (here, τ), only the intervals on each side of the 180°_y.

14.15.3 Hasn't something been simplified here?

The excitation by a 90° pulse to convert the initial longitudinal magnetization into a completely transverse magnetization, followed by a 180° a time τ later to convert the phases of the still transverse magnetization is obviously a specific instantiation of pulses. But we know that we can generate an RF excitation pulse of any angle α, and similarly we could apply a refocussing pulse of any angle α'. And we could apply many pulses in a row, and with arbitrary timings in between them. Aside from the question of why we might want to do that, is "What happens to the magnetization?". The general description of this process is not trivial and we will discuss this in Chapter 21, but the conceptual problem is something that you might want to start thinking about at this point. So consider Figure 14.15 and think about what would happen if the excitation pulse (Figure **??**) was *smaller* than 90°, say 45°. In that case, some of the initial magnetization would be put in the transverse plane and some would remain along the longitudinal axes. Specifically, the transverse component would be the *projection* (Section **??**) of the magnetization onto the transverse (xy) plane and the longitudinal component would be the projection of the magnetization onto the longitudinal (z) axis. Subsequent pulses, either further excitations or refocussing pulses, then tip *each of these components*, creating projections of these projections. And, of course, in the time between pulses, the longitudinal components relax according to T_1 while the transverse component decay according to T_2. As you can imagine, as the number of pulses is increased, the number of these separate projections multiply, the pattern of detectable transverse components becoming quite complicated. But there are some very good reasons for doing this, as we shall see.

Suggested reading