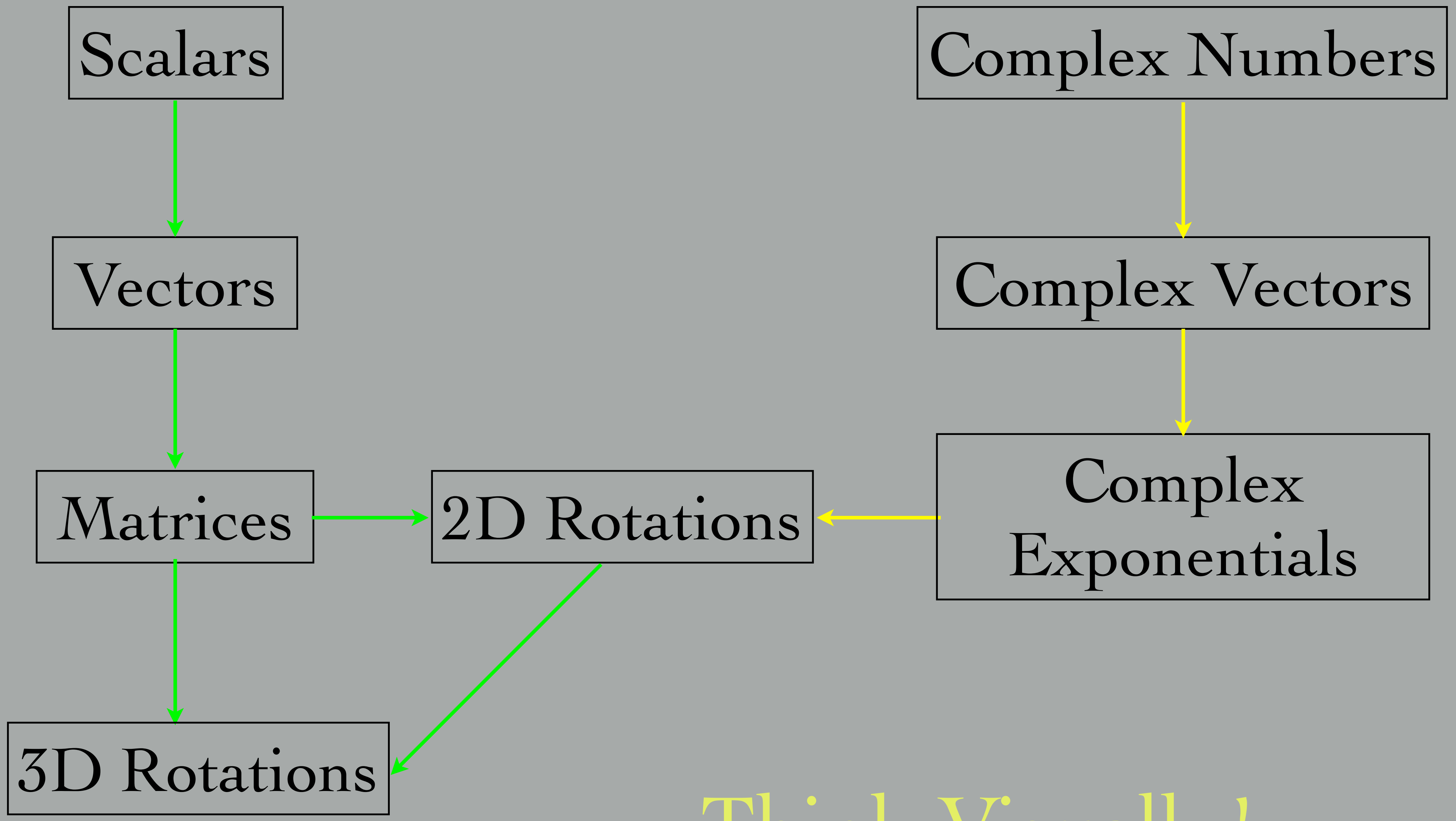
Lecture 3
Vectors, Matrices, and Complex Numbers

Lecture Summary



Think Visually!

Lecture Summary

Most of this chapter is just packaging and bookkeeping

Vectors

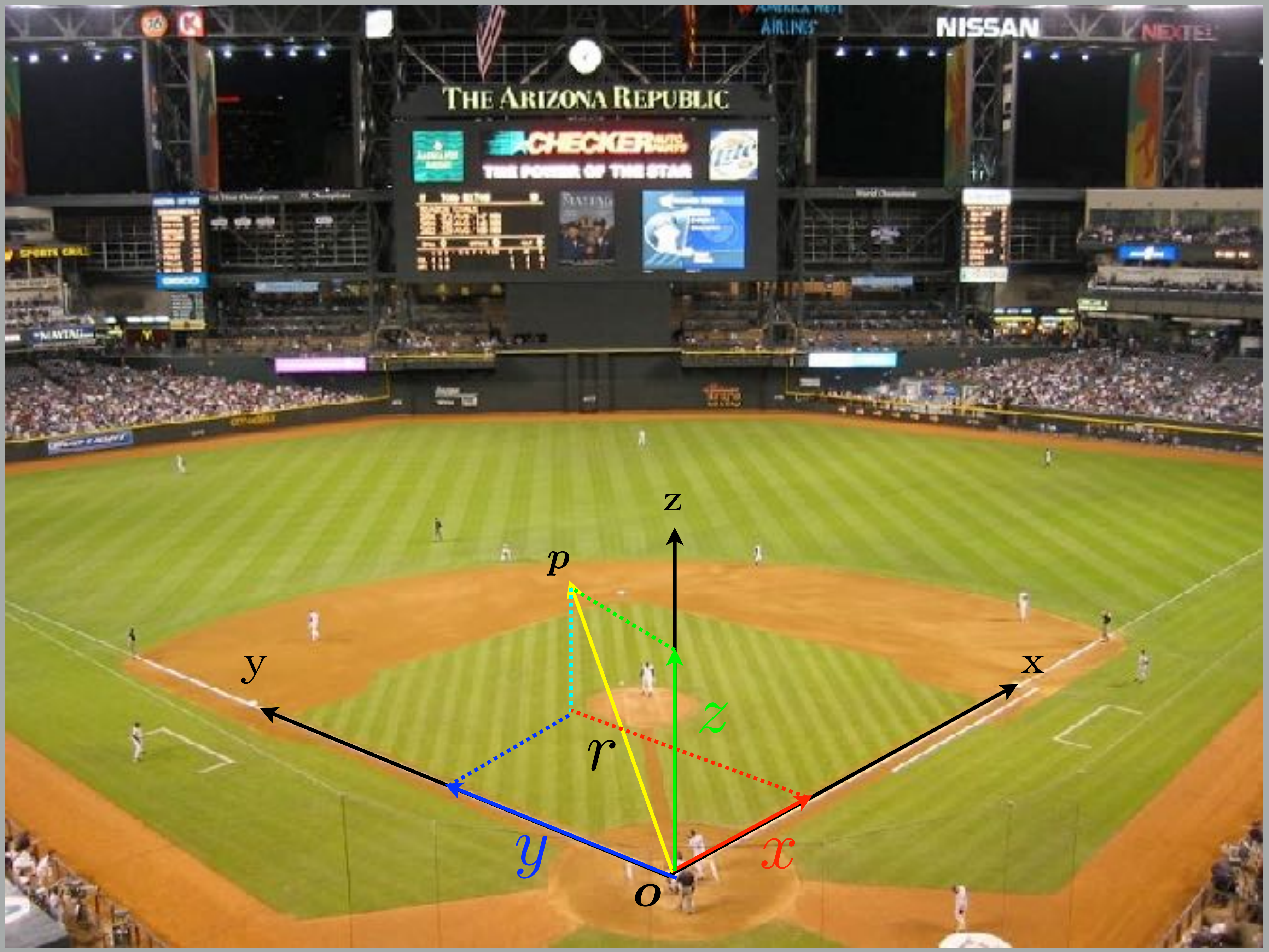
ball has a *direction* and a *velocity*

Cartesian coordinate system

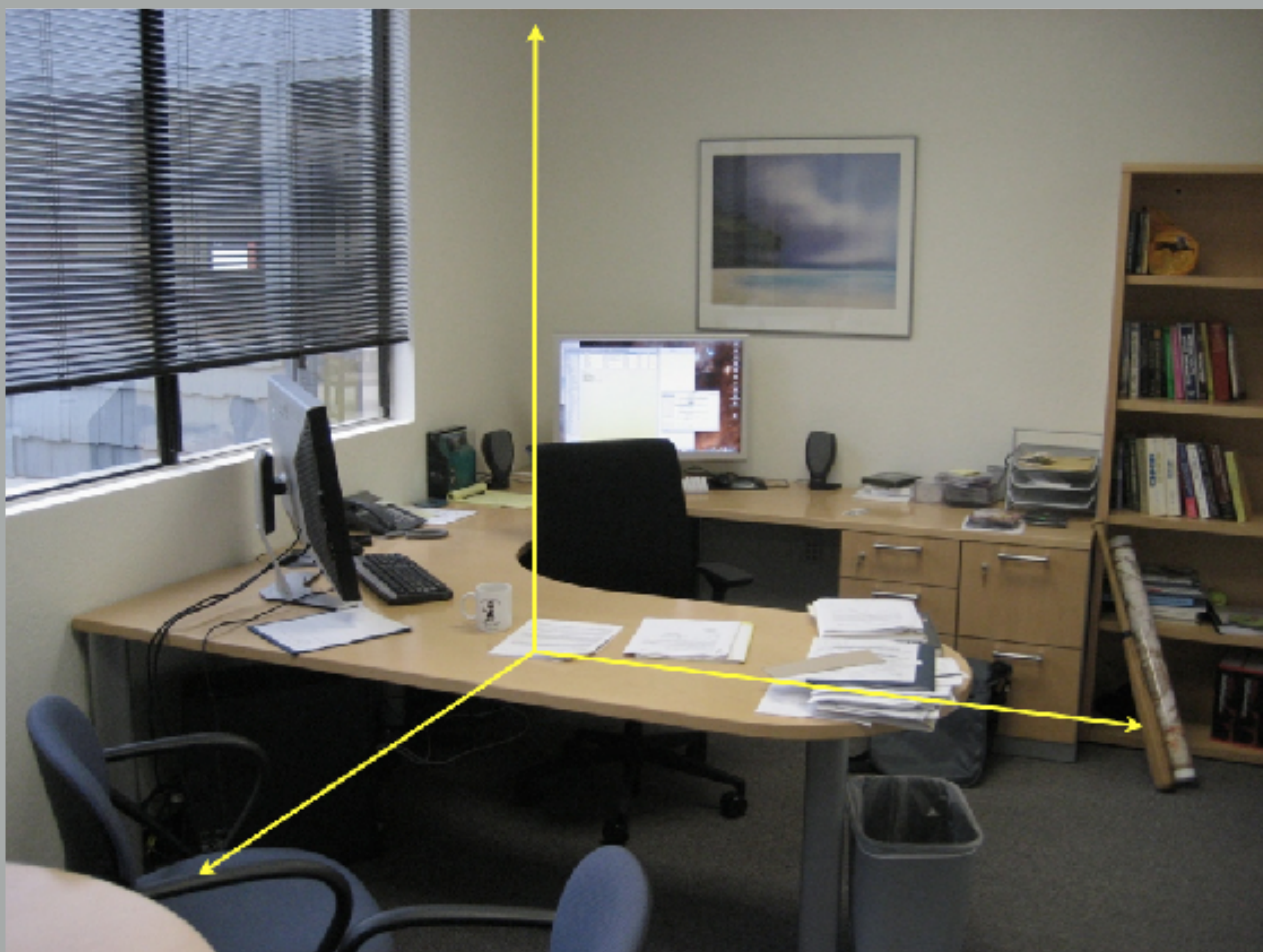
first, we need a coordinate system

Cartesian coordinate system

Vector in 3D



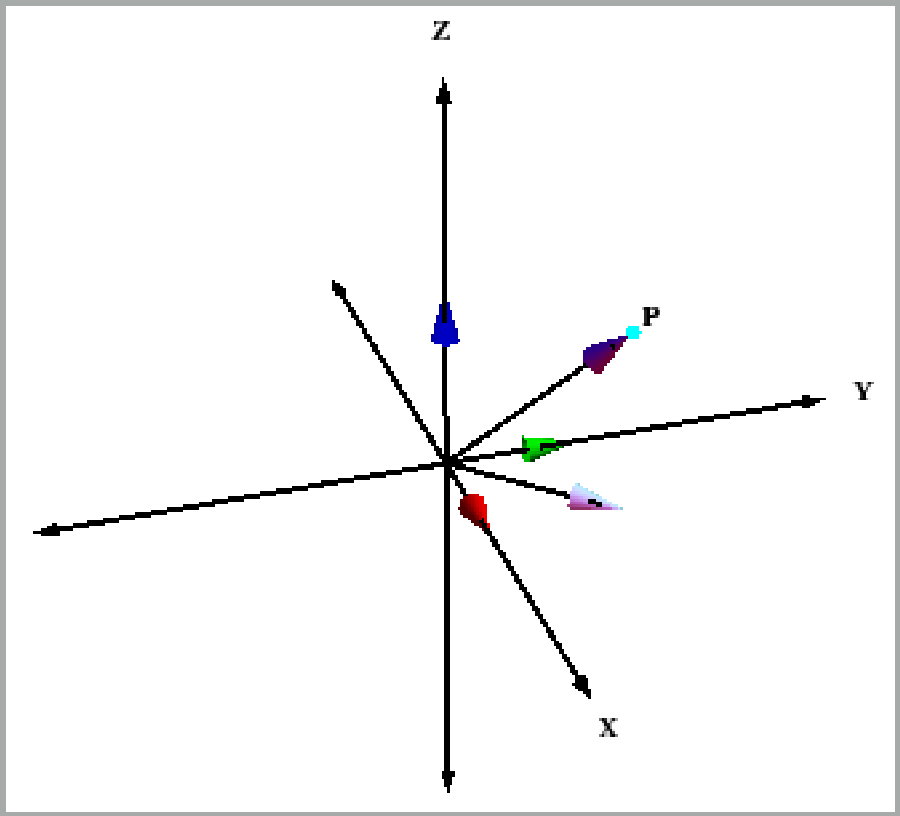
Coordinate Systems



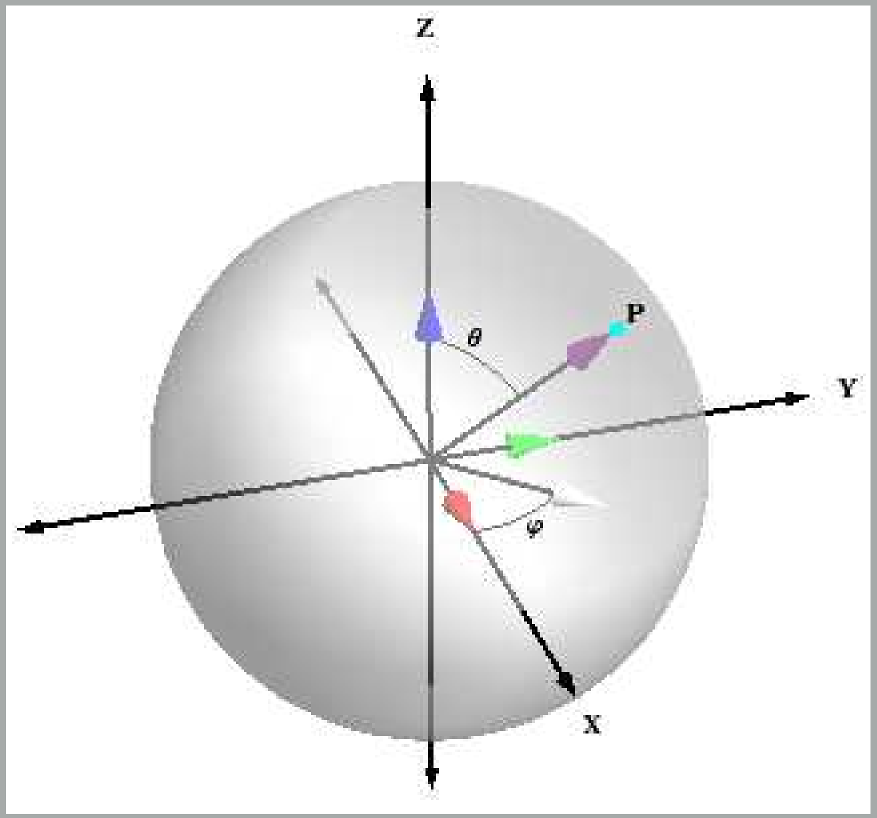
Cartesian

Spherical

Coordinate Systems

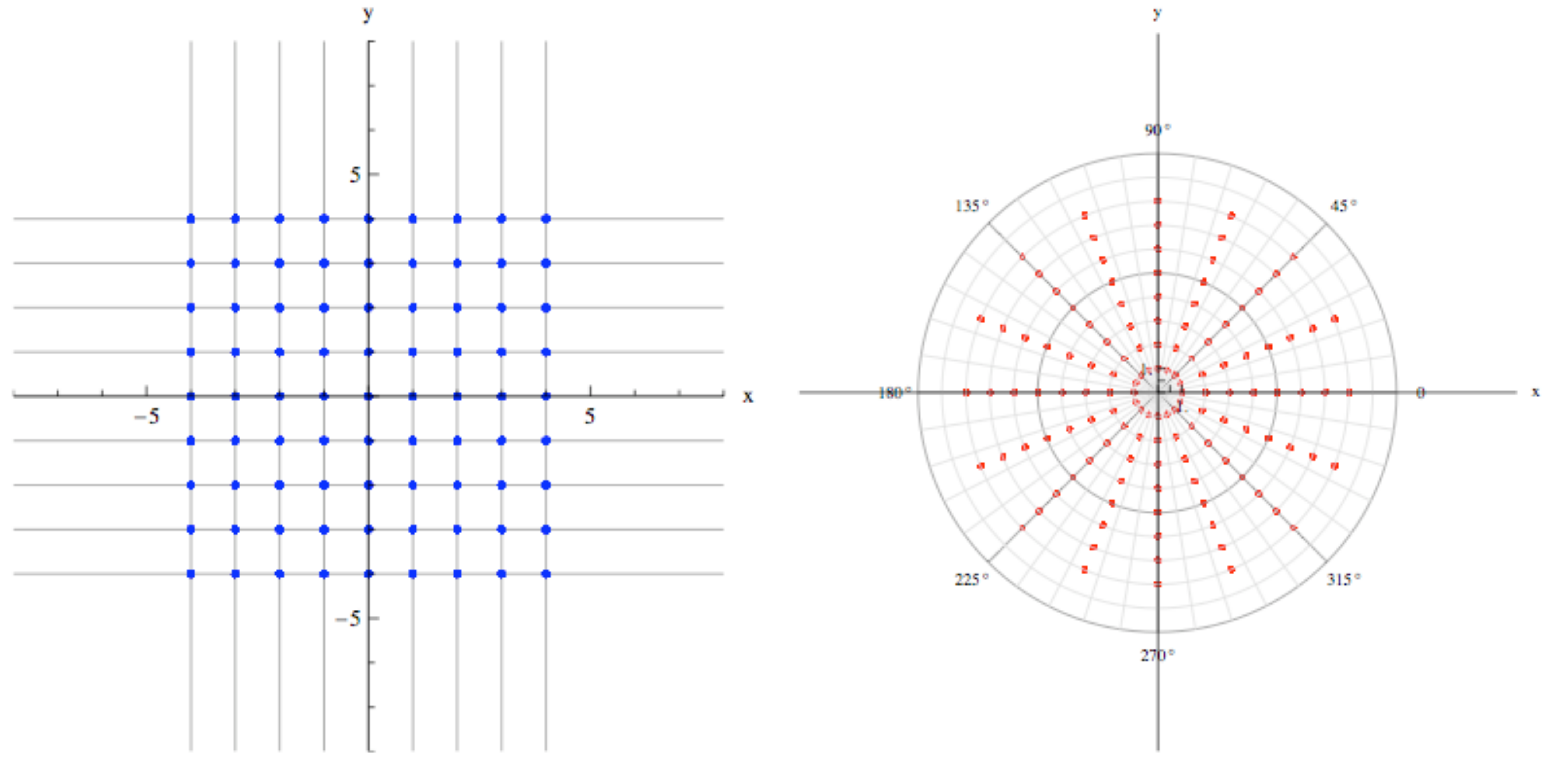


Cartesian



Spherical

Example: MRI



(a) Cartesian grid.

(b) Polar grid.

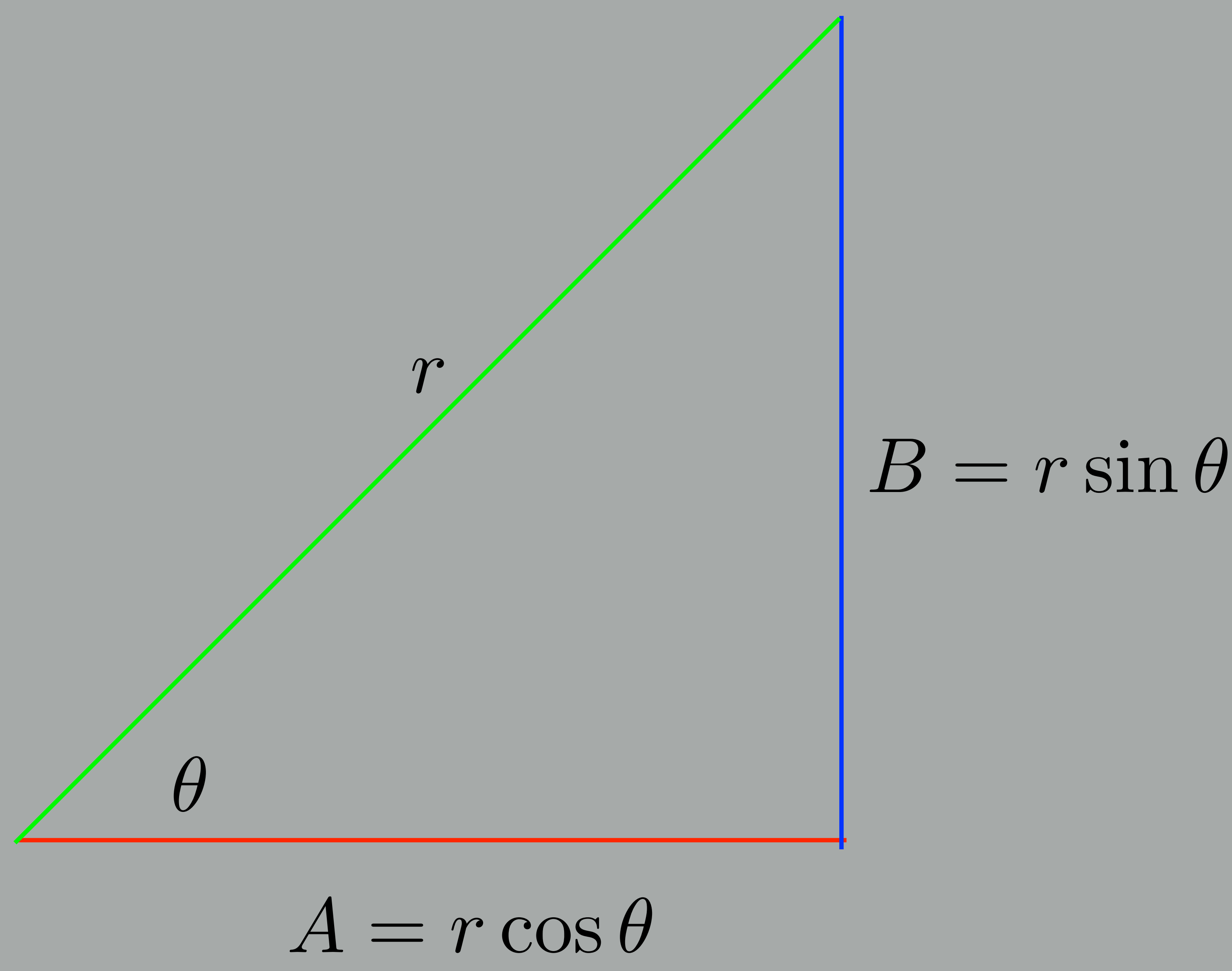
Echo planar

Spiral

Trigonometry

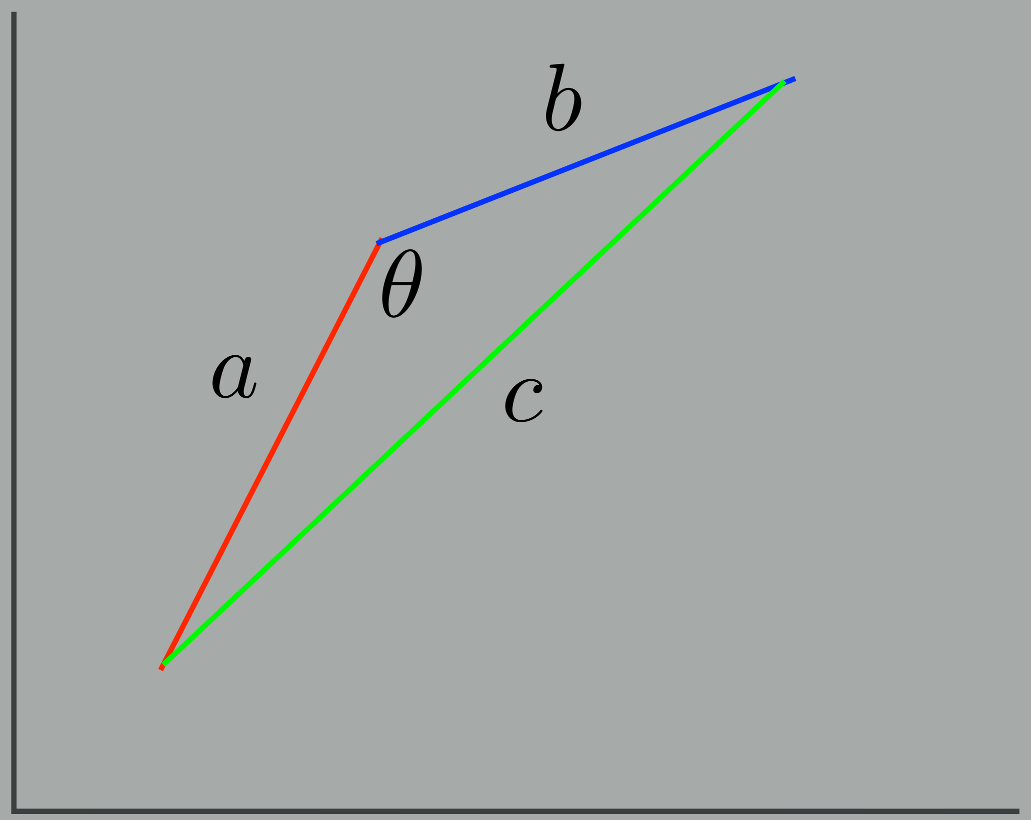
$$\cos\theta = A/r$$

$$\sin\theta = B/r$$



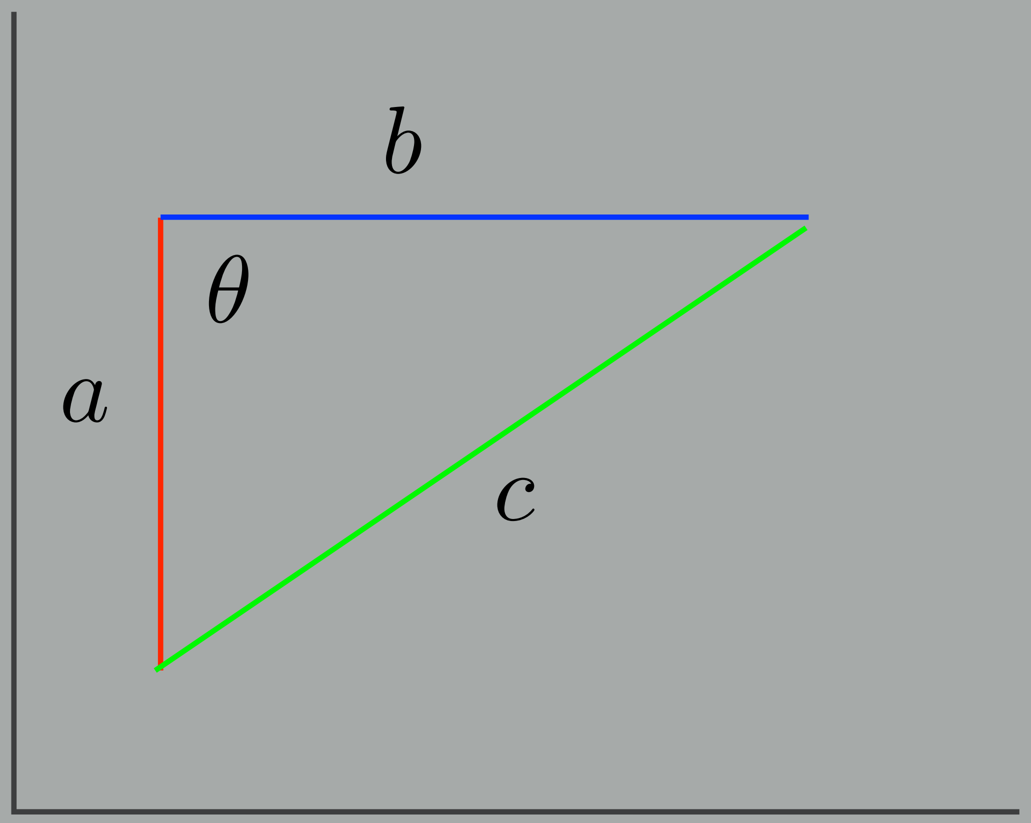
Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



Law of Cosines

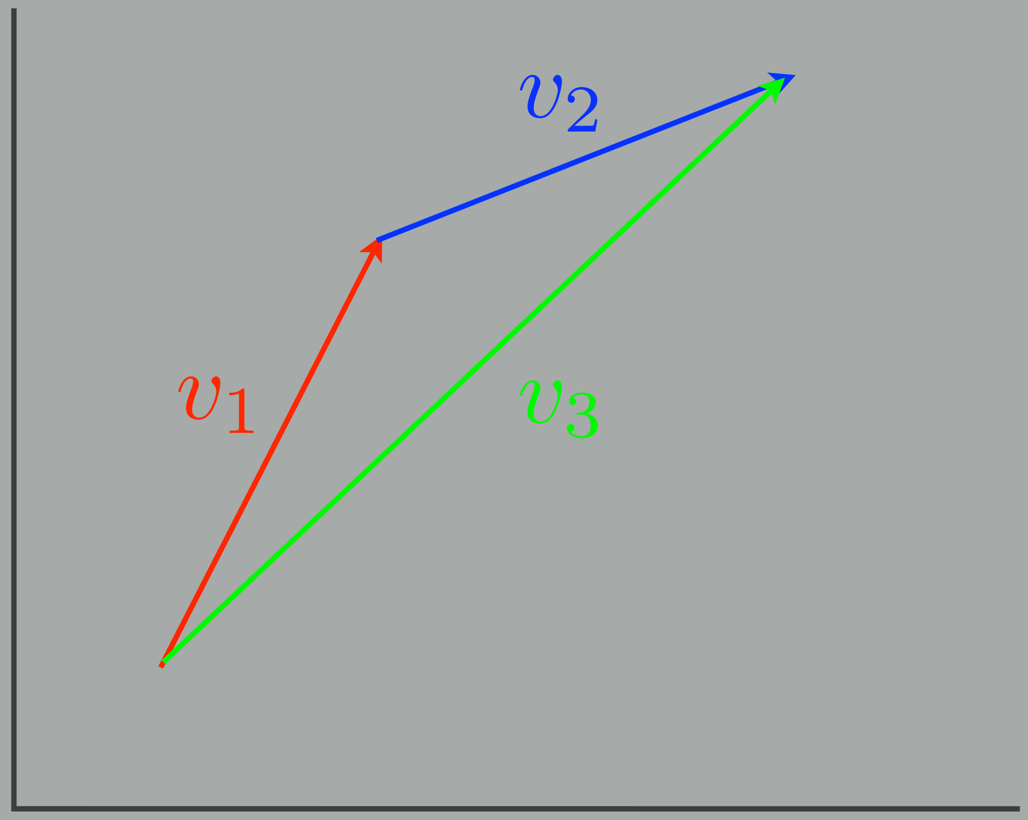
$$c^2 = a^2 + b^2 \quad \text{(ref: Pythagoras)}$$



For right triangle: $\cos 90^\circ = 0$

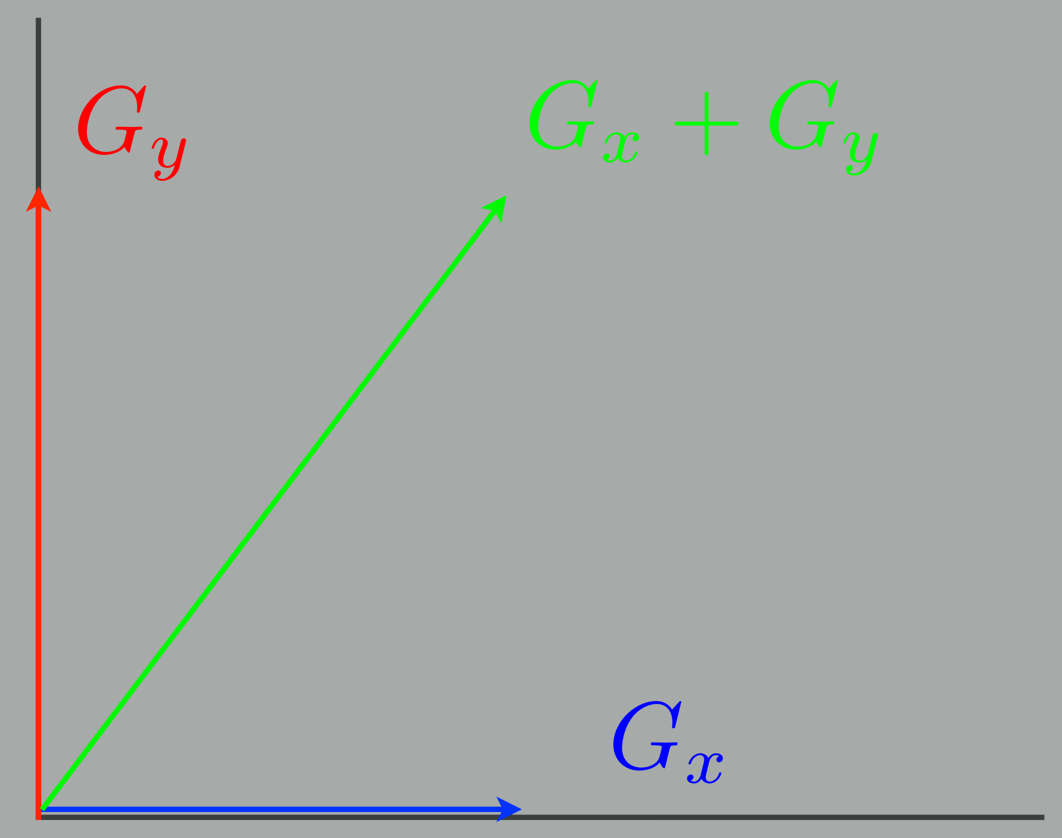
Vector Addition

$$\boldsymbol{v}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2$$



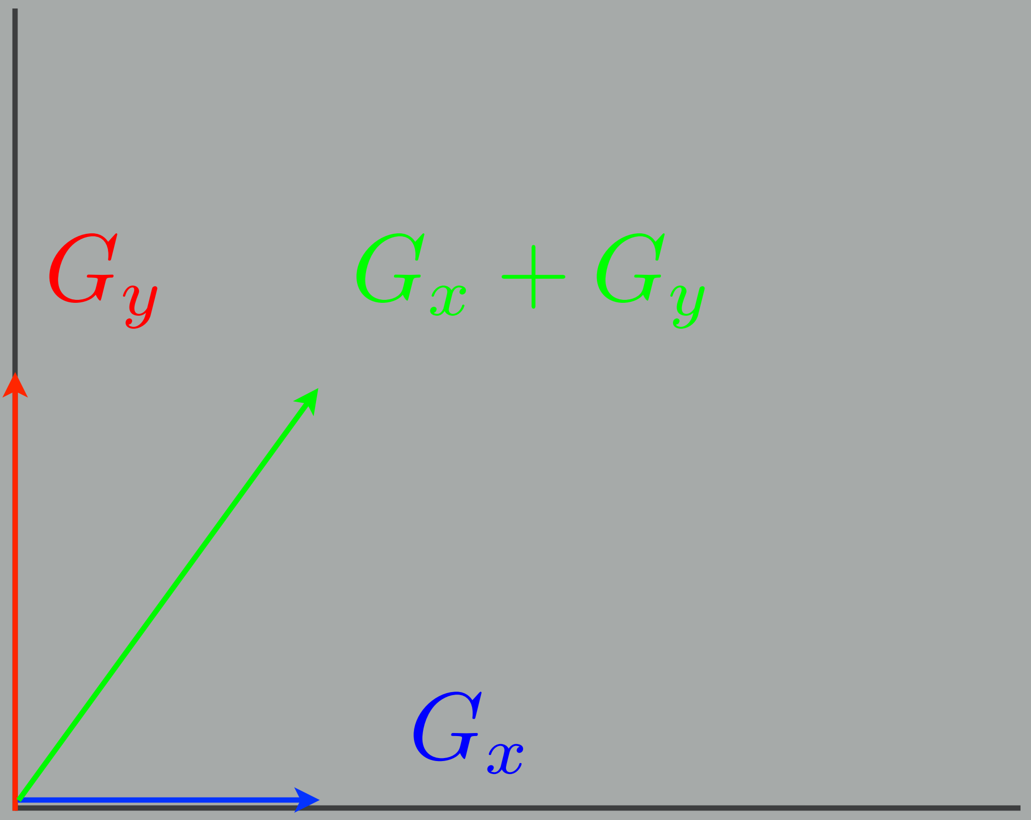
Example

MRI field gradient add like vectors

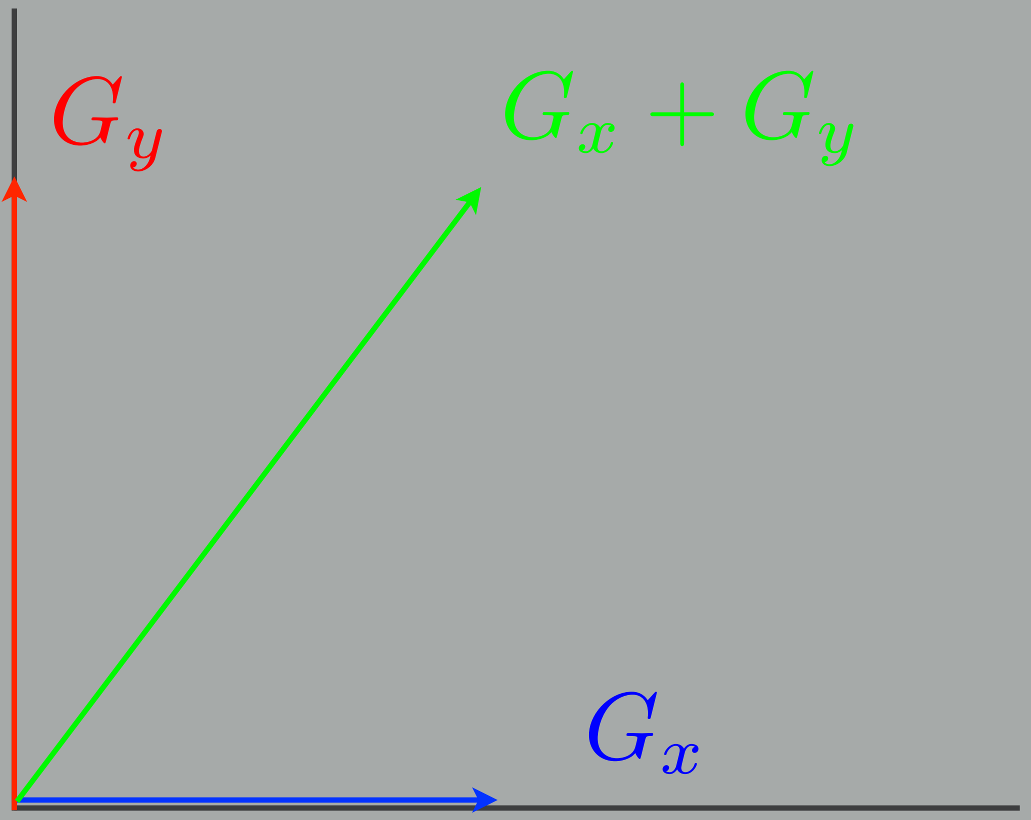


Example

Strength of applied gradients



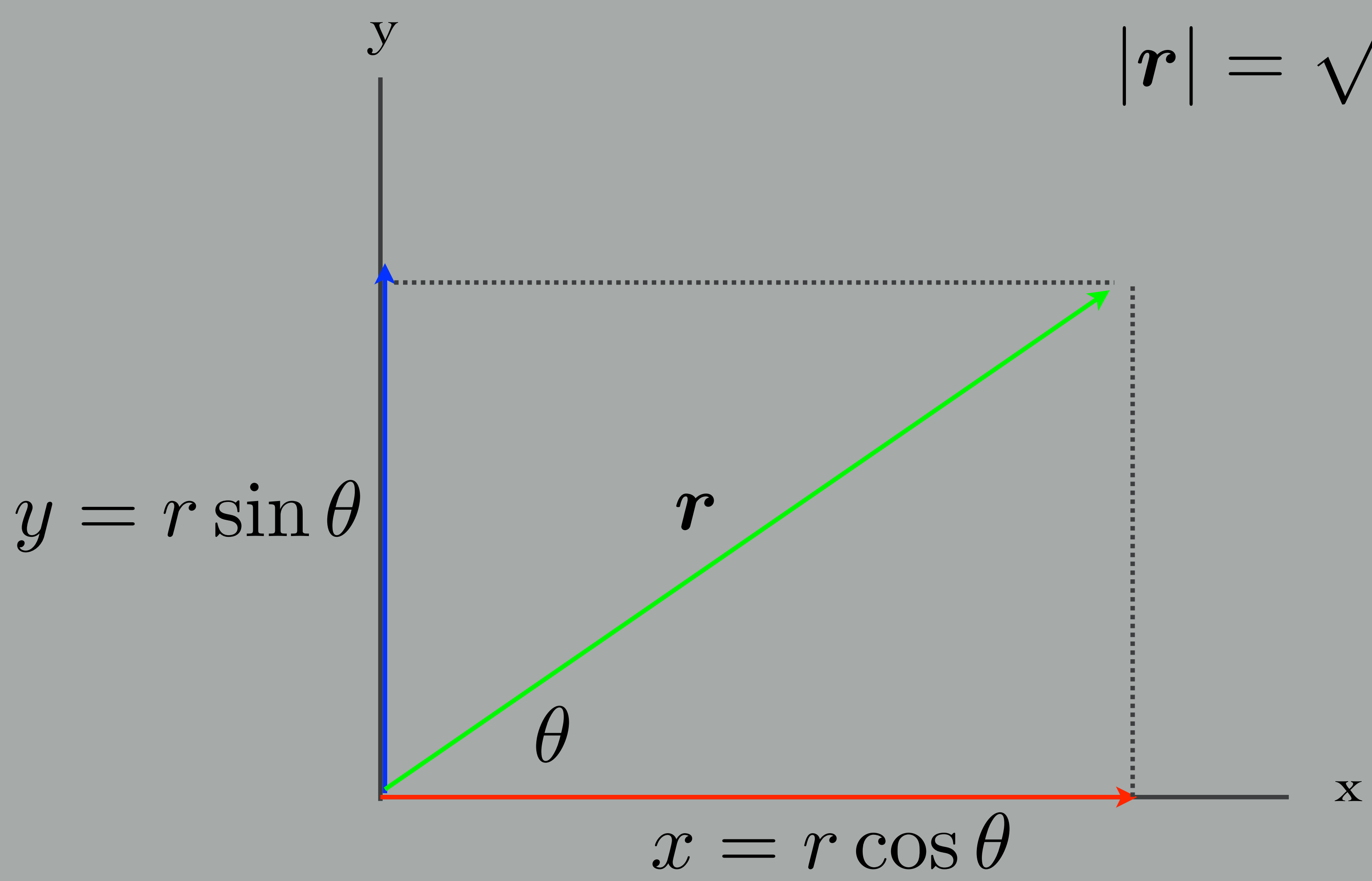
weaker



stronger

Vector Length (Orthogonal)

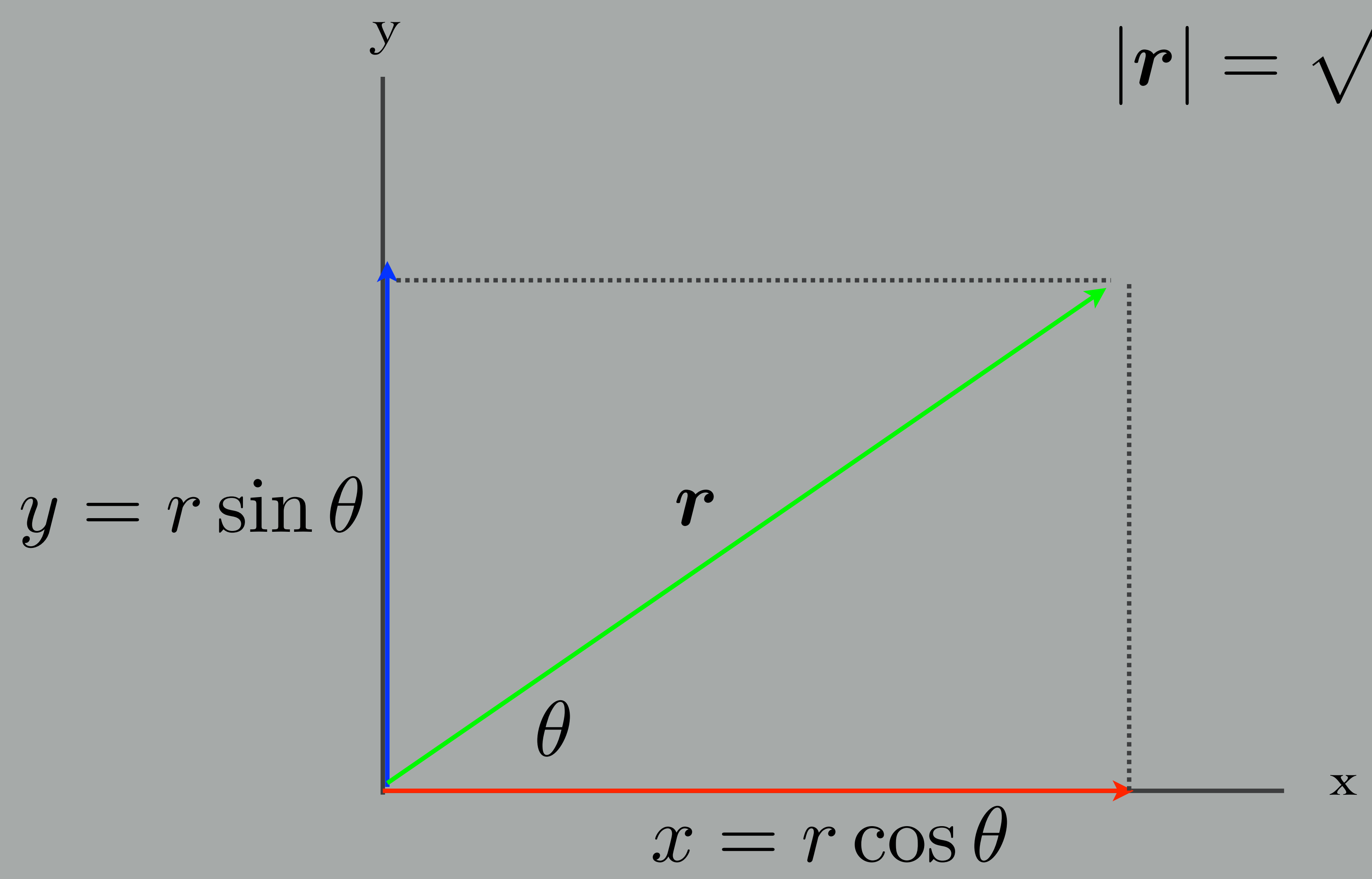
$$|\boldsymbol{r}| = \sqrt{x^2 + y^2}$$



$x^2 + y^2$

Vector Length (Orthogonal)

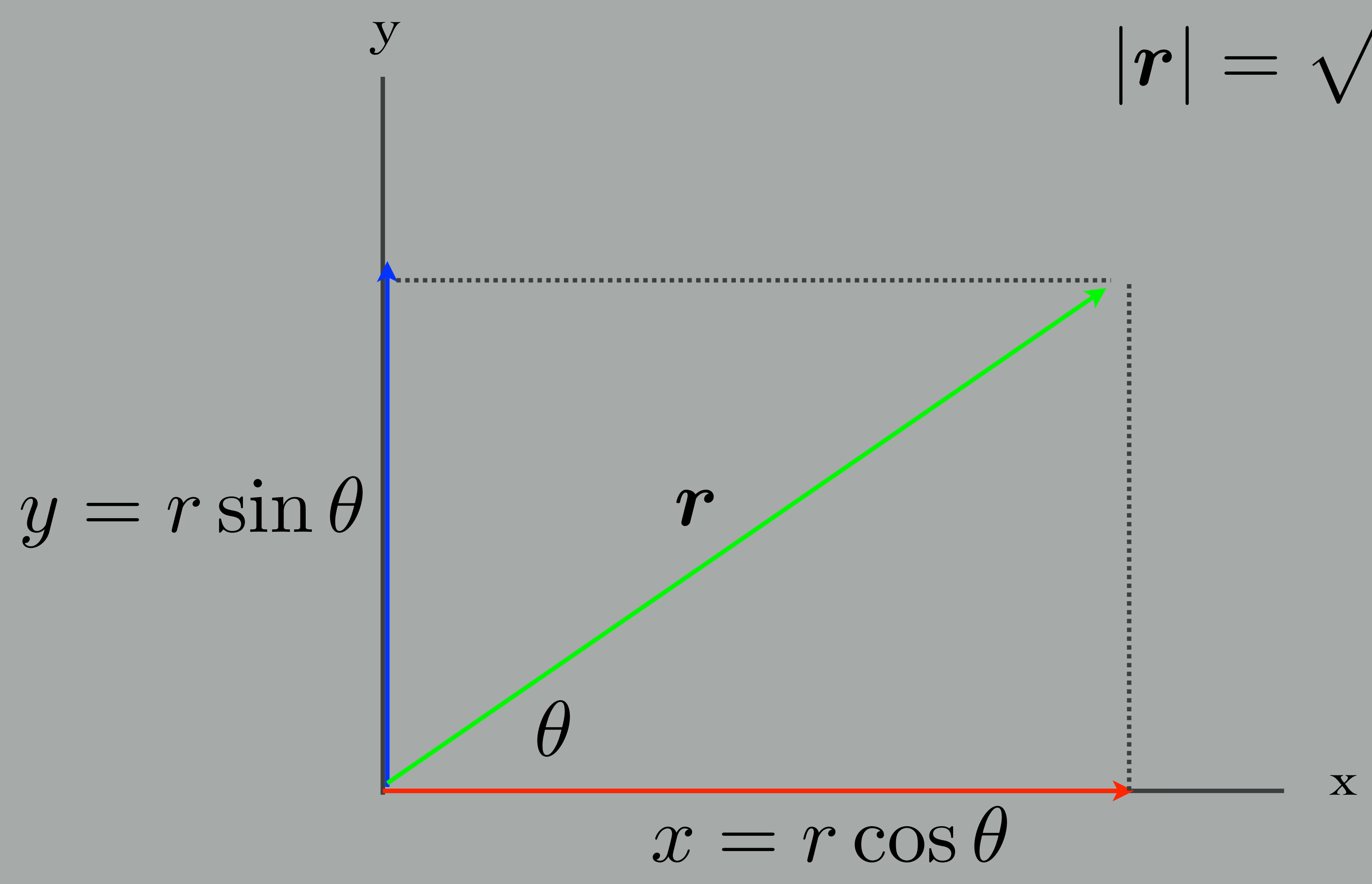
$$|\boldsymbol{r}| = \sqrt{x^2 + y^2}$$



$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

Vector Length (Orthogonal)

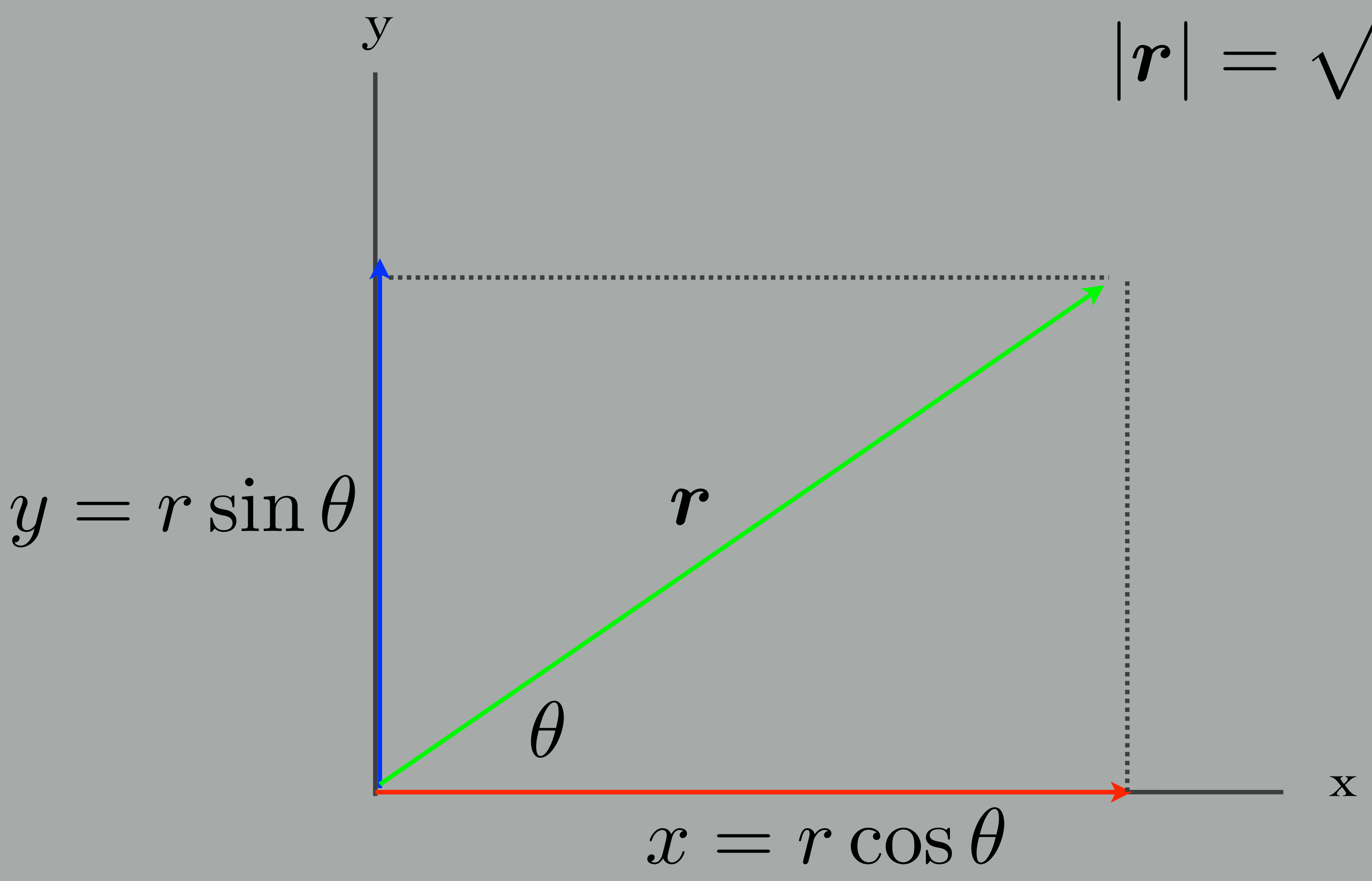
$$|\boldsymbol{r}| = \sqrt{x^2 + y^2}$$



$$x^2 + y^2 = r^2 \cos^2\theta + r^2 \sin^2\theta = r^2 \underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{1}$$

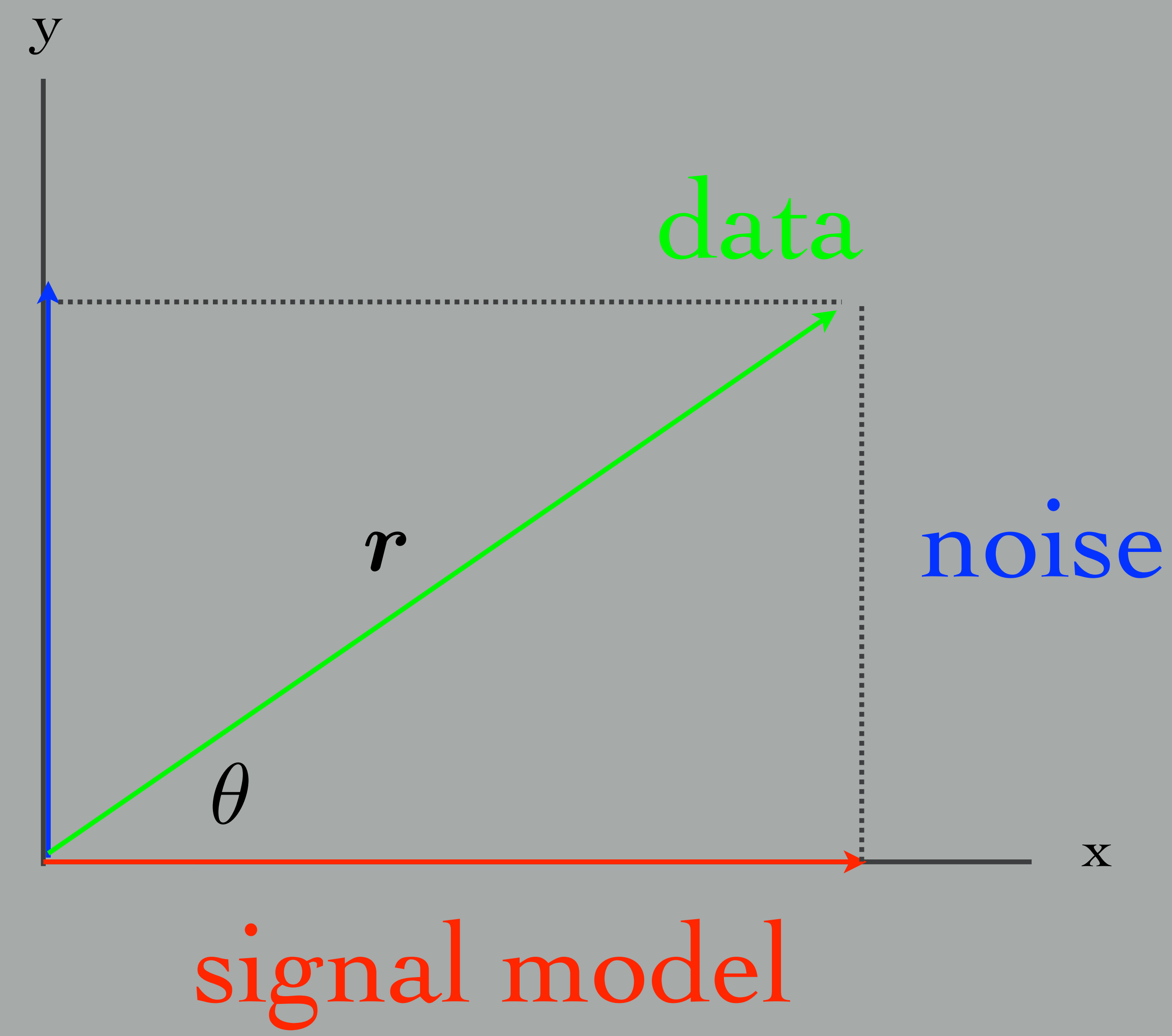
Vector Length (Orthogonal)

$$|\boldsymbol{r}| = \sqrt{x^2 + y^2}$$



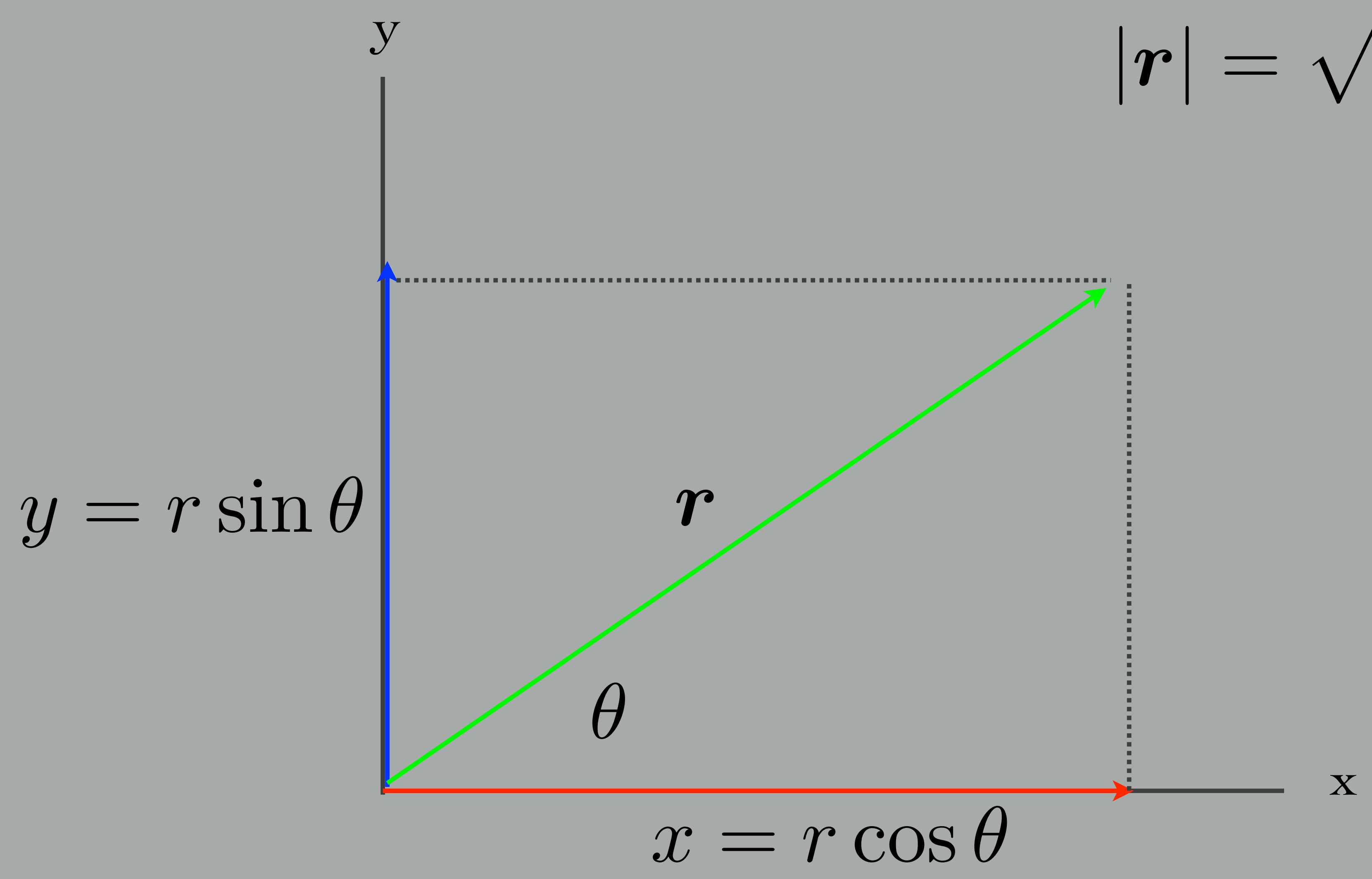
$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2\underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{1} = r^2$$

Example: Least Squares



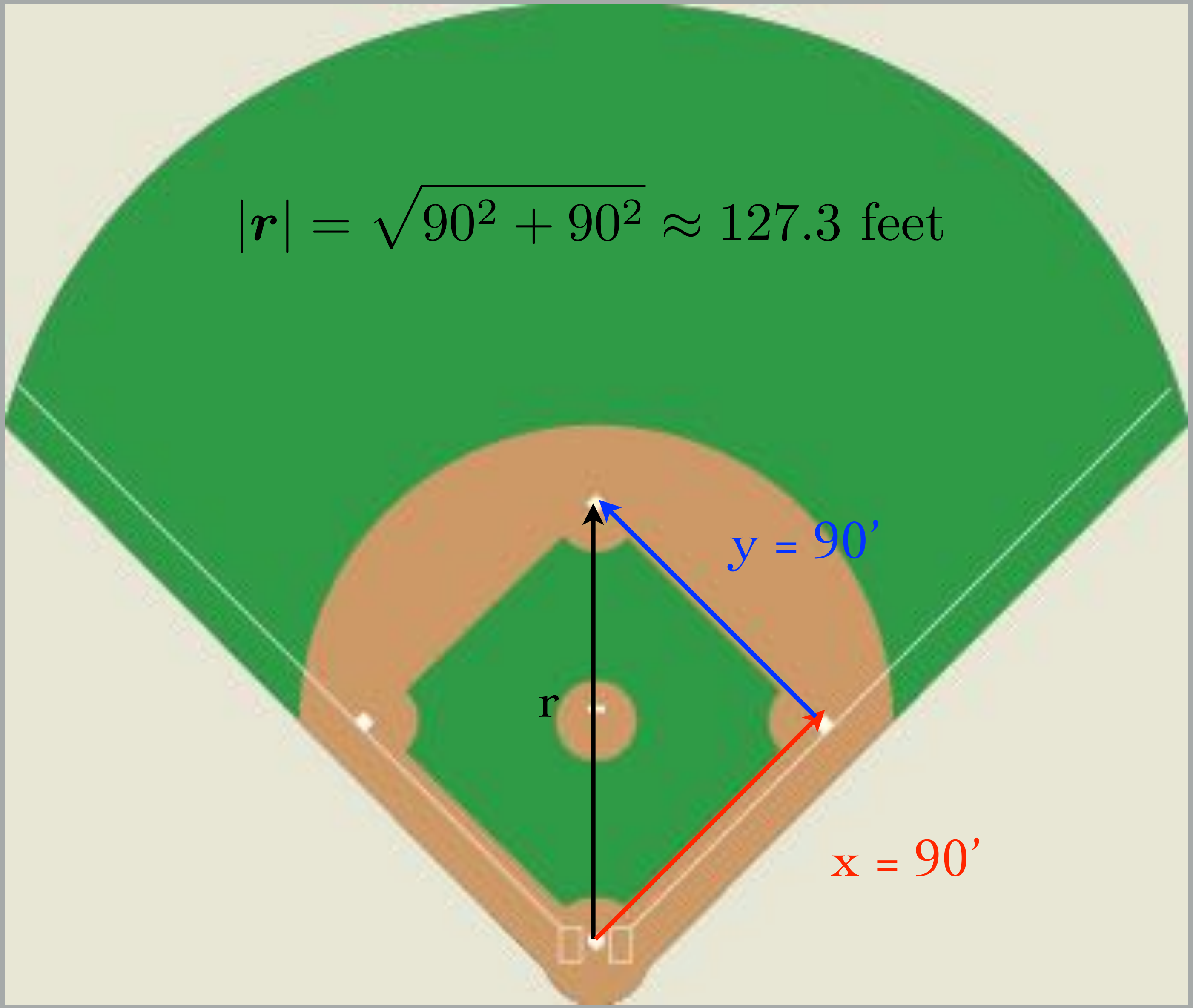
Vector Angle

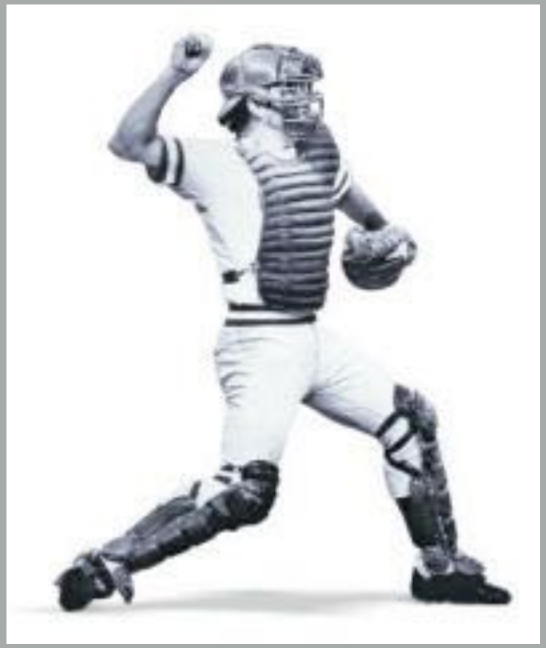
$$|\boldsymbol{r}| = \sqrt{x^2 + y^2}$$



$$\tan\theta = \frac{y}{x} \longrightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Vector Lengths





catcher throws to second

Vector Notation

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$ column vector

$$\boldsymbol{q} = (a\ ,\ b)$$ row vector

$$\boldsymbol{r}^t = (x\ ,\ y)$$ vector *transpose*
(exchange rows & columns)

often written in general form with row subscript

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

Vector and Equations

the line

$$y = \alpha + \beta x$$

Vector and Equations

the line

$$y = \alpha + \beta x$$

Write in vector form:

$$\boldsymbol{a} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \boldsymbol{f} = \begin{pmatrix} 1 \\ x \end{pmatrix}$$

Vector and Equations

the line

$$y = \alpha + \beta x$$

Write in vector form:

$$\boldsymbol{a} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \boldsymbol{f} = \begin{pmatrix} 1 \\ x \end{pmatrix}$$

"amplitudes" "model functions"

Vector and Equations

the line

$$y = \alpha + \beta x$$

Write in vector form:

$$\boldsymbol{a} = \begin{pmatrix} \boxed{\alpha} \\ \beta \end{pmatrix} \qquad \boldsymbol{f} = \begin{pmatrix} \boxed{1} \\ x \end{pmatrix}$$

Vector and Equations

the line

$$y = \alpha + \beta x$$

Write in vector form:

$$\boldsymbol{a} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \boldsymbol{f} = \begin{pmatrix} 1 \\ x \end{pmatrix}$$

Vector Dot Product

$$\boldsymbol{a} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \boldsymbol{f} = \begin{pmatrix} 1 \\ x \end{pmatrix}$$

Vector multiplication rule:
Row elements times column elements and add

$$y = \begin{pmatrix} \boxed{\alpha} & \beta \end{pmatrix} \begin{pmatrix} \boxed{1} \\ x \end{pmatrix}$$

Vector Dot Product

$$\boldsymbol{a} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \boldsymbol{f} = \begin{pmatrix} 1 \\ x \end{pmatrix}$$

Vector multiplication rule:
Row elements times column elements and add

$$y = \begin{pmatrix} \boxed{\alpha} & \beta \end{pmatrix} \begin{pmatrix} \boxed{1} \\ x \end{pmatrix} = \boxed{\alpha}$$

Vector Dot Product

$$\boldsymbol{a} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \boldsymbol{f} = \begin{pmatrix} 1 \\ x \end{pmatrix}$$

Vector multiplication rule:
Row elements times column elements and add

$$y = \begin{pmatrix} \boxed{\alpha} & \beta \end{pmatrix} \begin{pmatrix} \boxed{1} \\ x \end{pmatrix} = \boxed{\alpha} +$$

Vector Dot Product

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Vector Dot Product

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Vector multiplication rule:
Row elements times column elements and add

$$y = \begin{pmatrix} \boxed{\alpha} & \boxed{\beta} \end{pmatrix} \begin{pmatrix} \boxed{1} \\ \boxed{x} \end{pmatrix} = \boxed{\alpha} + \boxed{\beta x}$$

columns here .... must equal *rows* here

Vector Dot Product

$$\boldsymbol{a} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \boldsymbol{f} = \begin{pmatrix} 1 \\ x \end{pmatrix}$$

Vector multiplication rule:
Row elements times column elements and add

$$y = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \alpha + \beta x$$

$$\boxed{y = \boldsymbol{a} \cdot \boldsymbol{f} = \boldsymbol{a}^t \boldsymbol{f}}$$

Vector and Equations

$\boxed{\text{Vector Dot Product}}$

$$y = \boldsymbol{a} \cdot \boldsymbol{f} = \boxed{\alpha 1} + \boxed{\beta x}$$

$$y = \sum_{i=1}^{2} a_i f_i$$

Vector dot product

linear equation

$$f = \quad ax + by + cz$$

Write in vector form:

$$\boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$f = \underbrace{\begin{pmatrix} x & y & z \end{pmatrix}}_{\mathbf{u}^t} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{v}} = \mathbf{u}^t\mathbf{v}$$

Vector dot product

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Write in vector form:

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$$f = \underbrace{\begin{pmatrix} x & \boxed{y} & z \end{pmatrix}}_{\mathbf{u}^t} \underbrace{\begin{pmatrix} x \\ \boxed{y} \\ z \end{pmatrix}}_{\mathbf{v}} = \mathbf{u}^t\mathbf{v}$$

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$$f = \underbrace{\begin{pmatrix} x & y & z \end{pmatrix}}_{\mathbf{u}^t} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{v}} = \mathbf{u}^t\mathbf{v}$$

Vector Length, revisited

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$ column vector

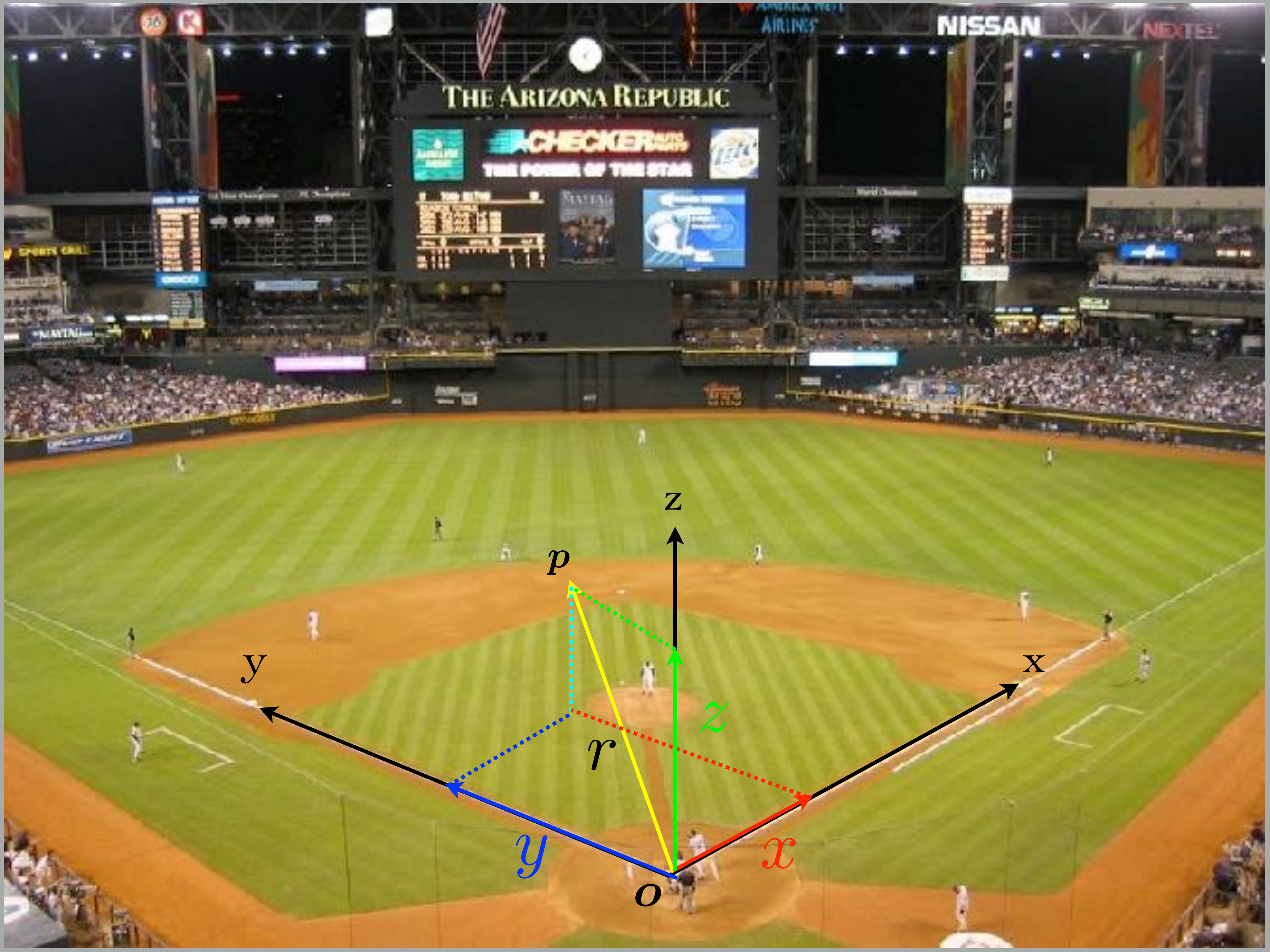
$$\boldsymbol{r}^t = (x \ , \ y)$$ vector *transpose*

$$\boldsymbol{r}^t \boldsymbol{r} = (x \ , \ y) \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$$

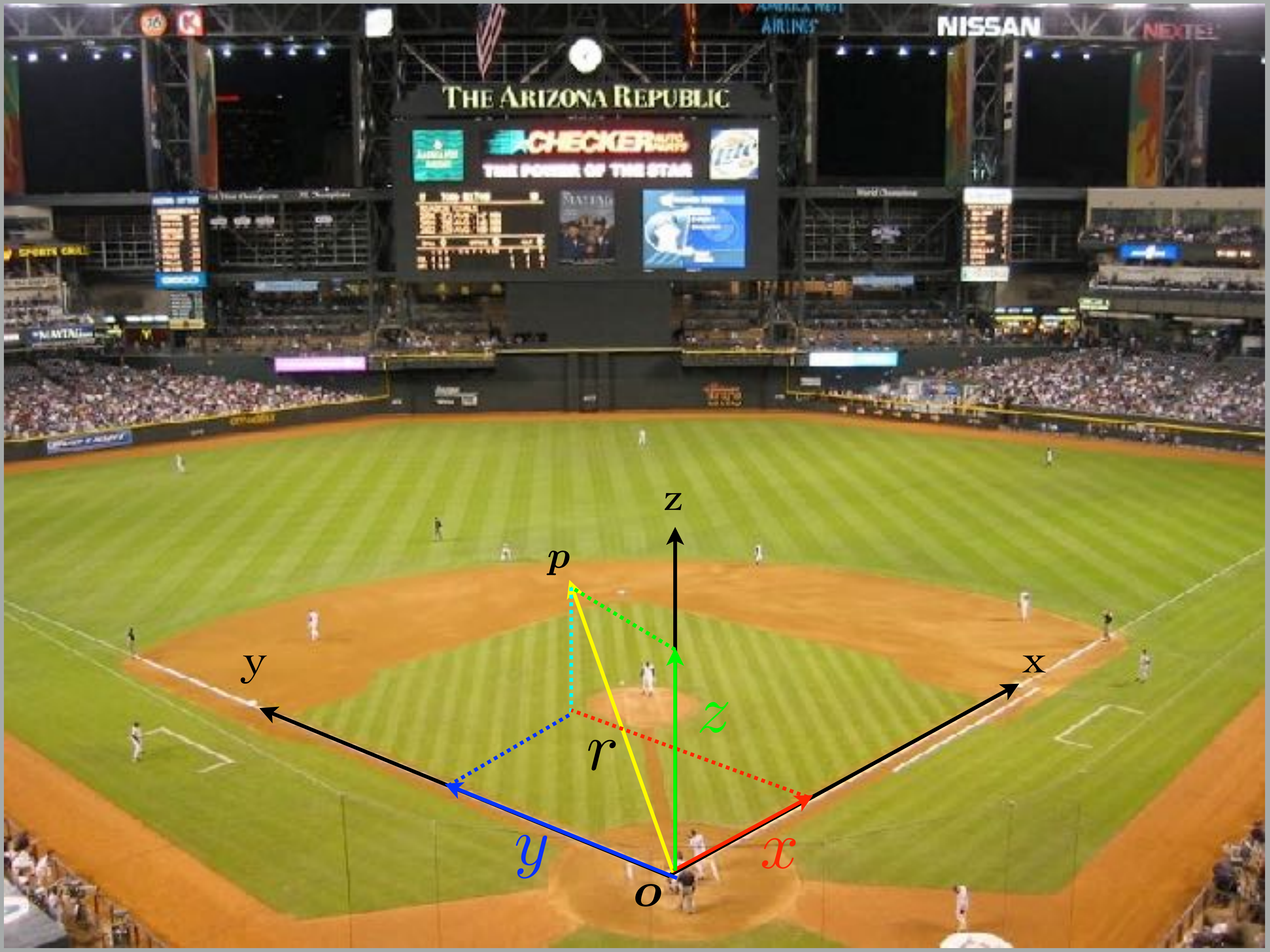
"therefore"

$$\therefore \boxed{\|\boldsymbol{r}\| = \sqrt{\boldsymbol{r}^t \boldsymbol{r}}}$$

Vector in 3D



Vector Length



$$\|\boldsymbol{r}\| = \sqrt{x^2 + y^2 + z^2} = \sqrt{\boldsymbol{r}^t \boldsymbol{r}}$$

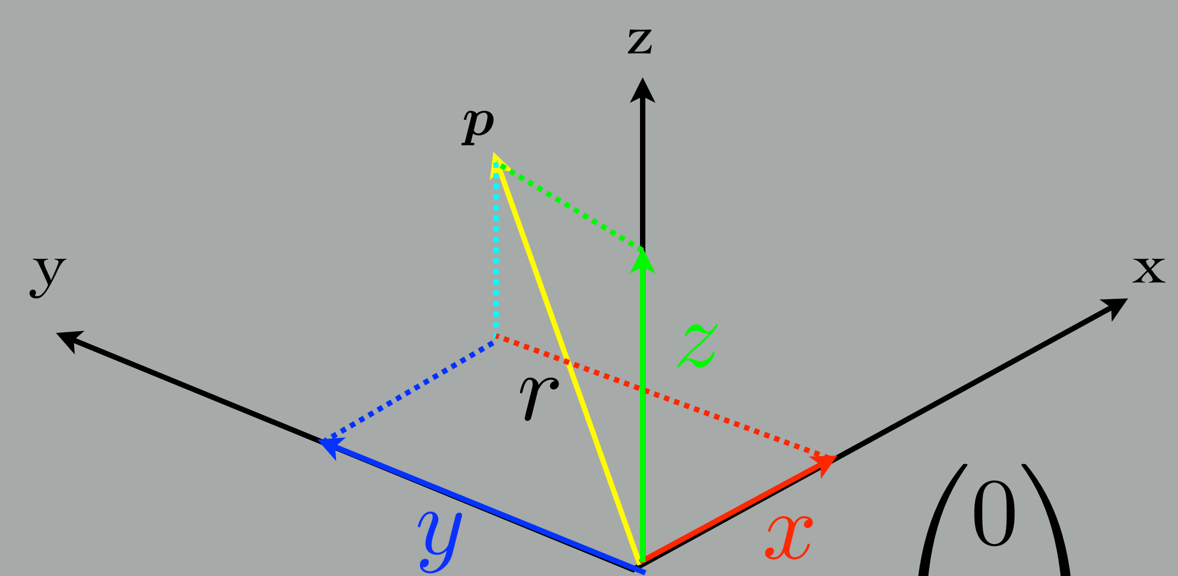
Vectors in 3D

Position vector

$$\boldsymbol{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Velocity vector

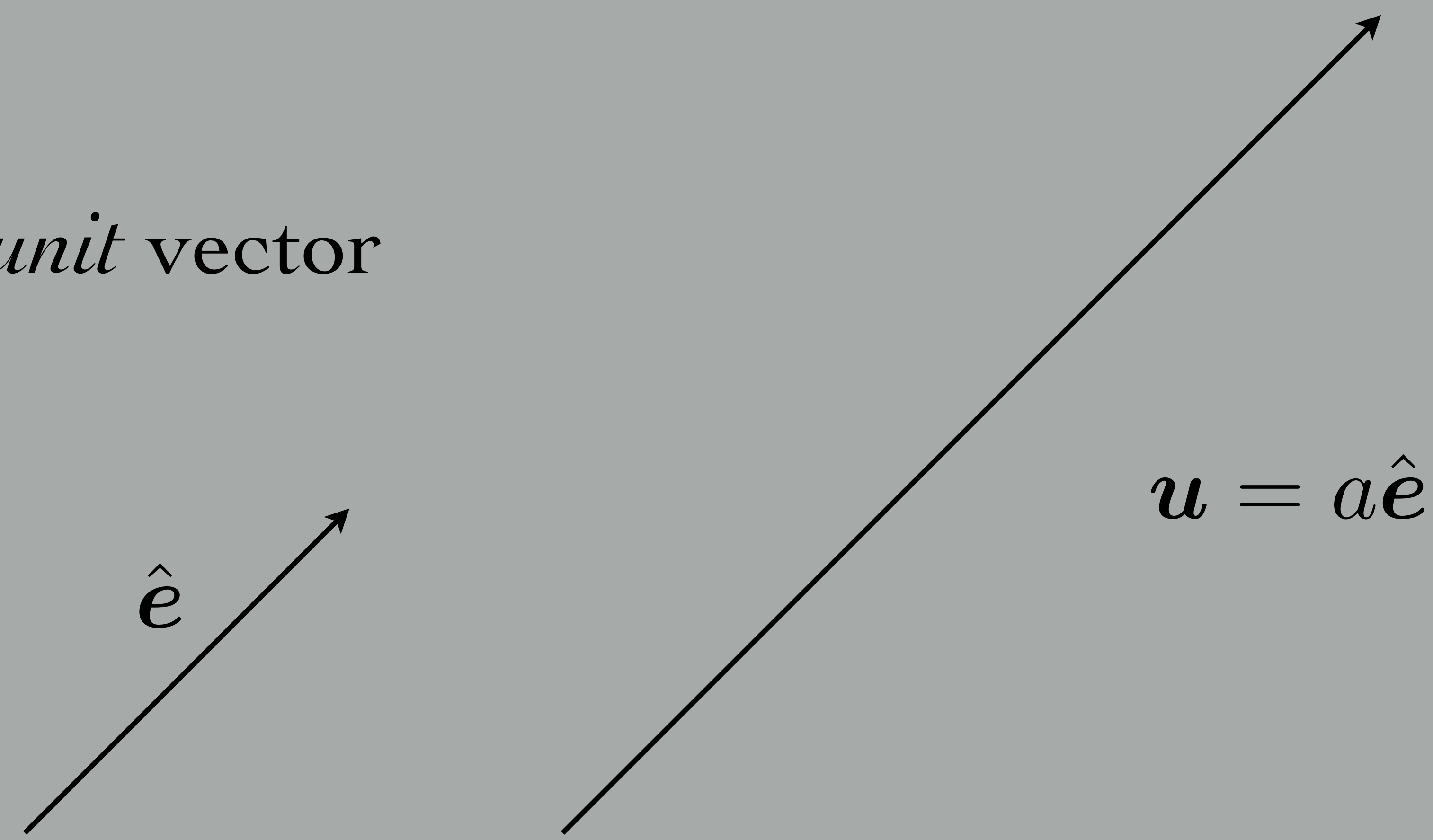
$$\boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$



$$\boldsymbol{o} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \text{origin}$$

Vector-Scalar Multiplication

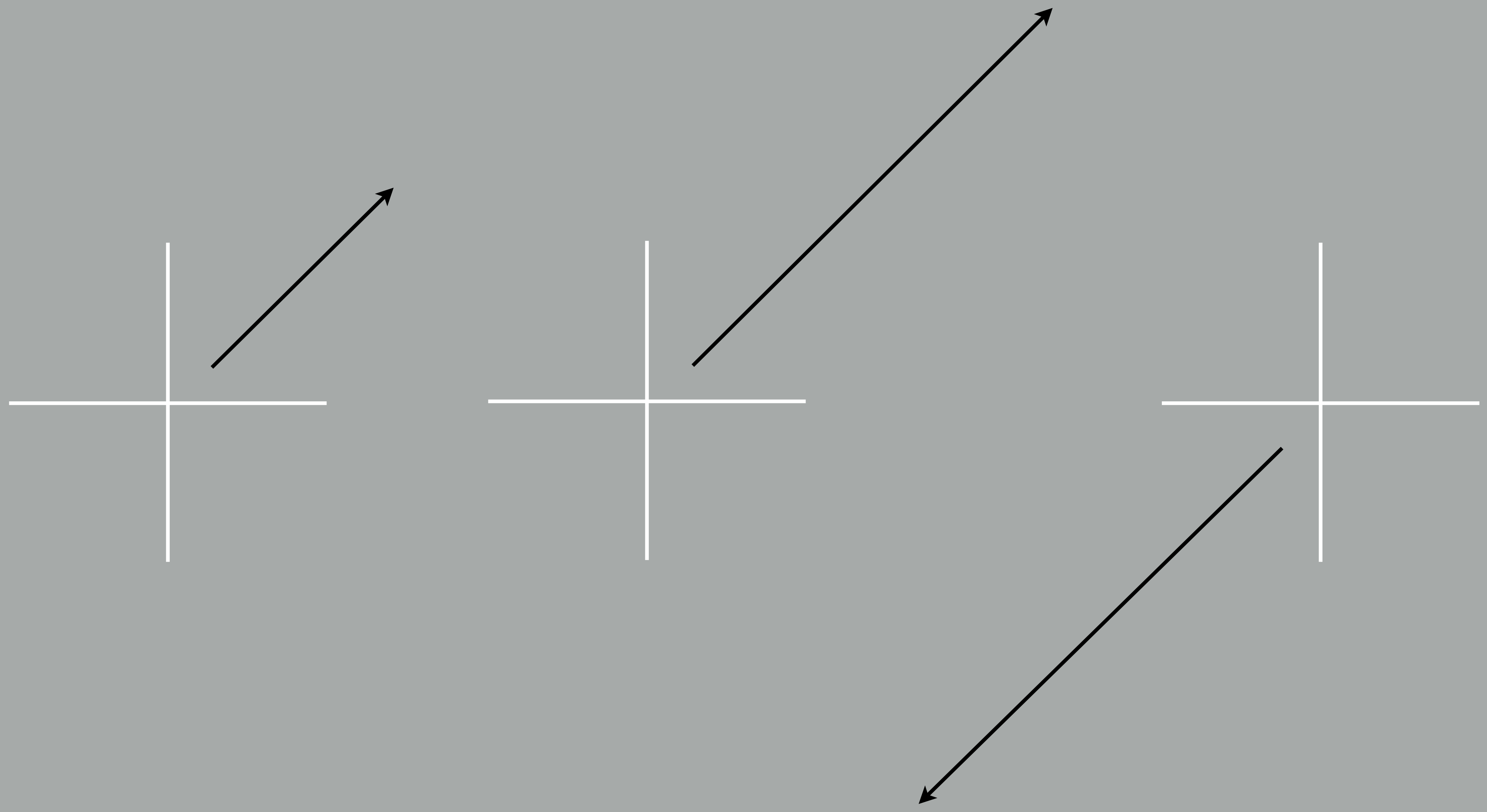
the *unit* vector



$$\|\hat{\boldsymbol{e}}\| = 1$$

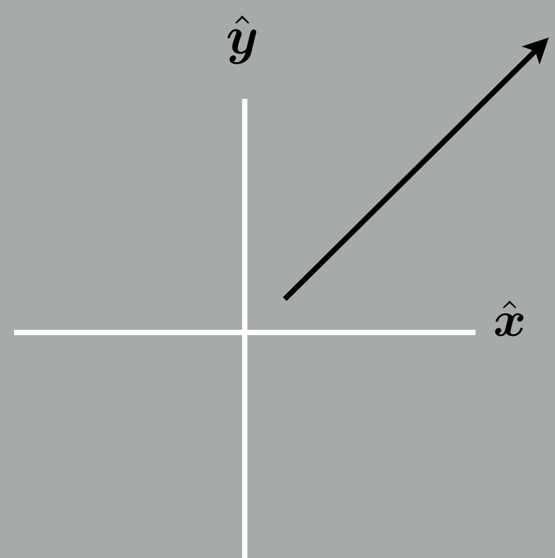
$$\|\boldsymbol{u}\| = a\|\hat{\boldsymbol{e}}\| = a$$

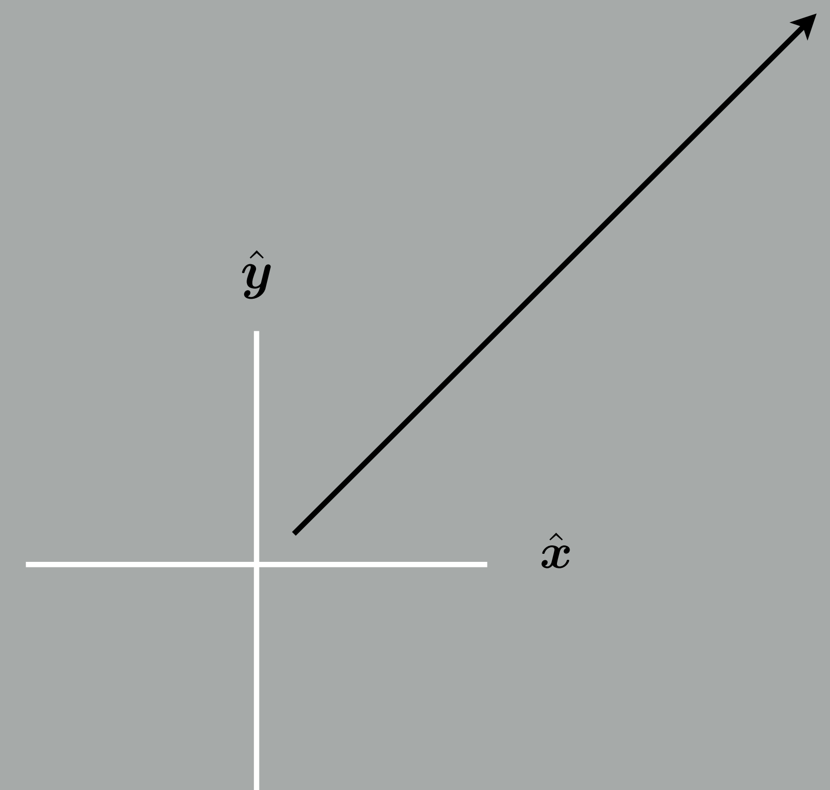
Vector-Scalar Multiplication



$\hat{\boldsymbol{v}}$ $\qquad$ $\boldsymbol{u} = 2\hat{\boldsymbol{v}}$ $\qquad$ $\boldsymbol{u} = -2\hat{\boldsymbol{v}}$

Example: Diffusion weighting





$\hat{\boldsymbol{g}}$

(the direction)

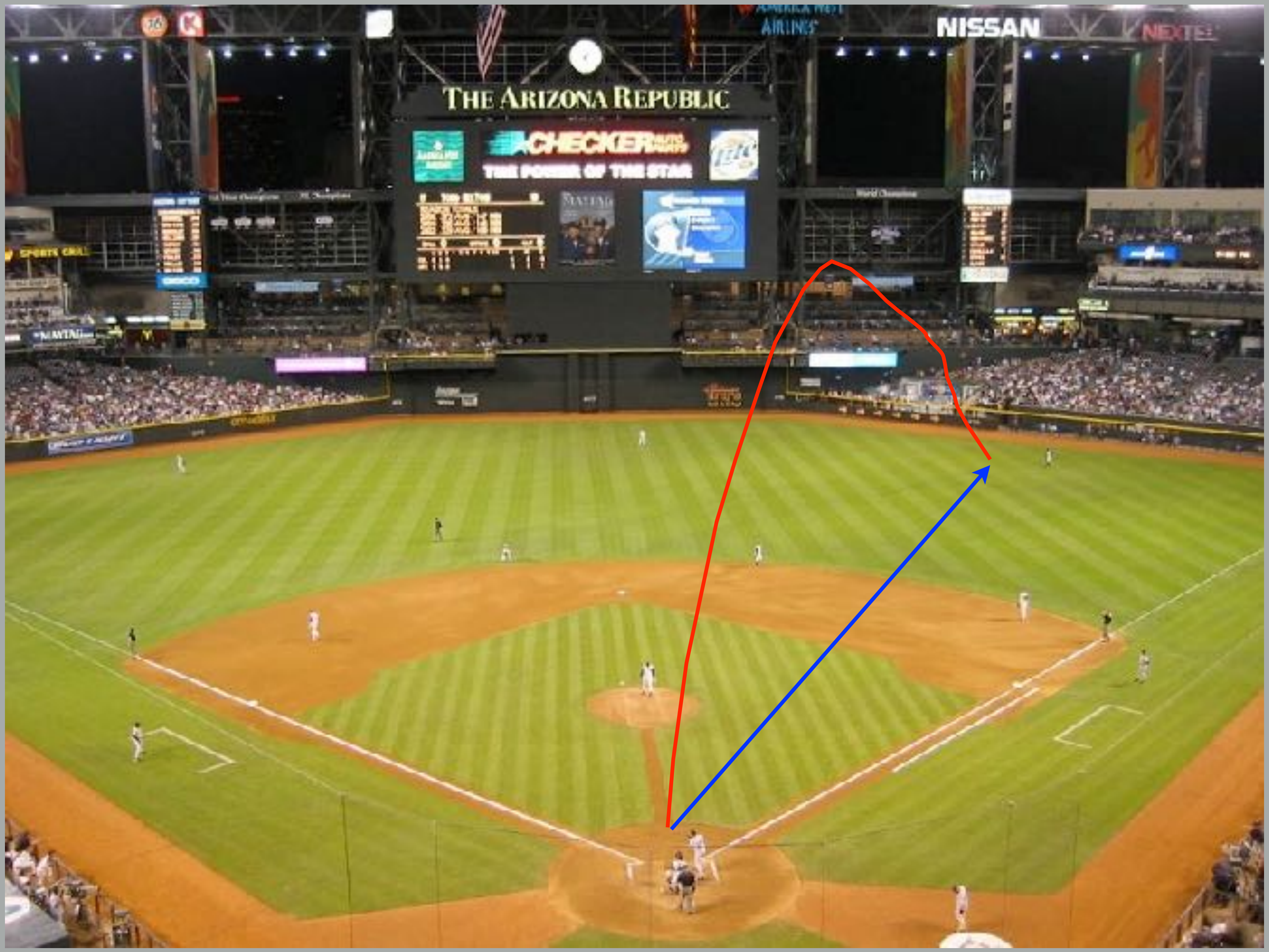
$\boldsymbol{g} = b\hat{\boldsymbol{g}}$

(the weighting)

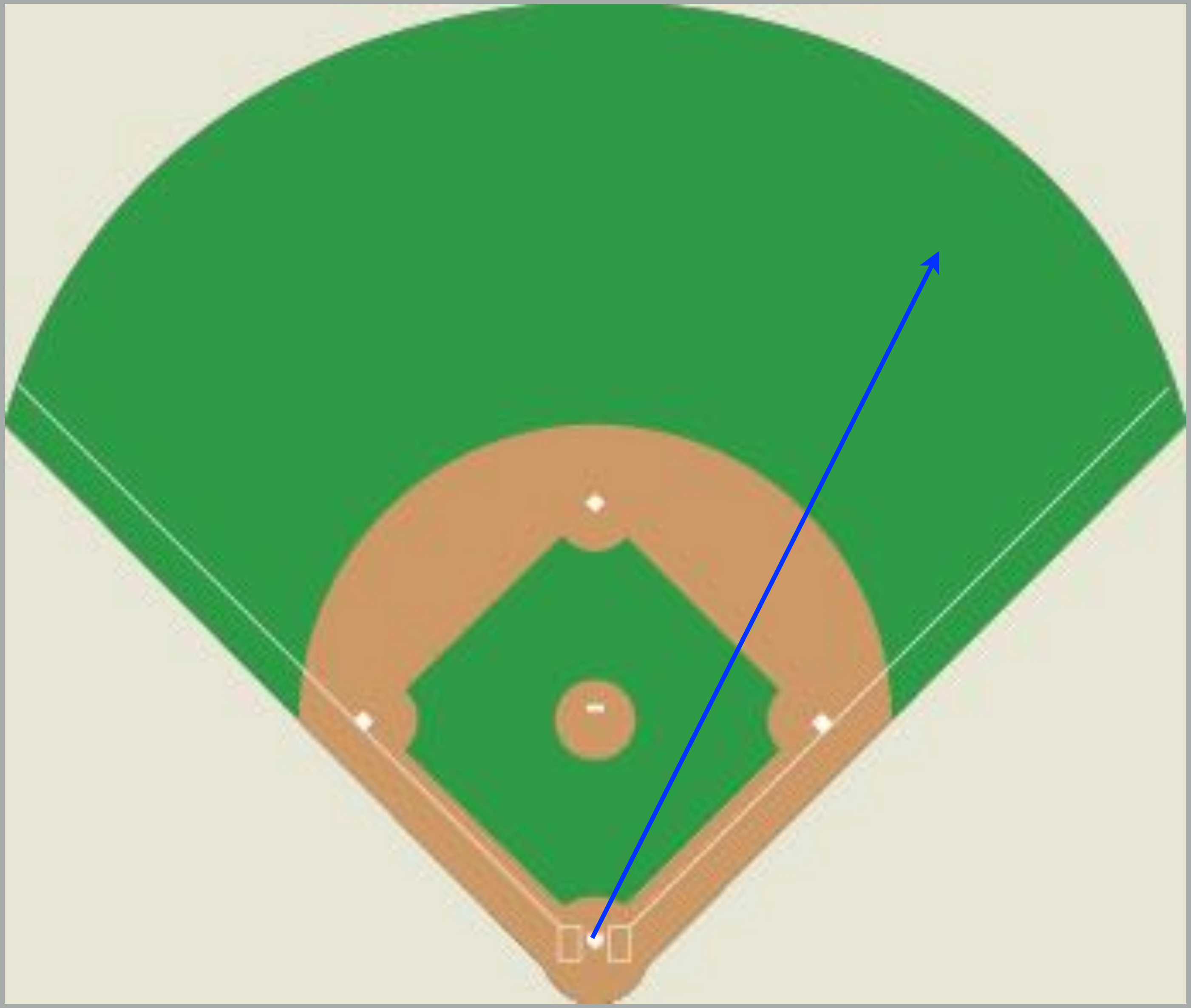
Projections

Projections

Projections

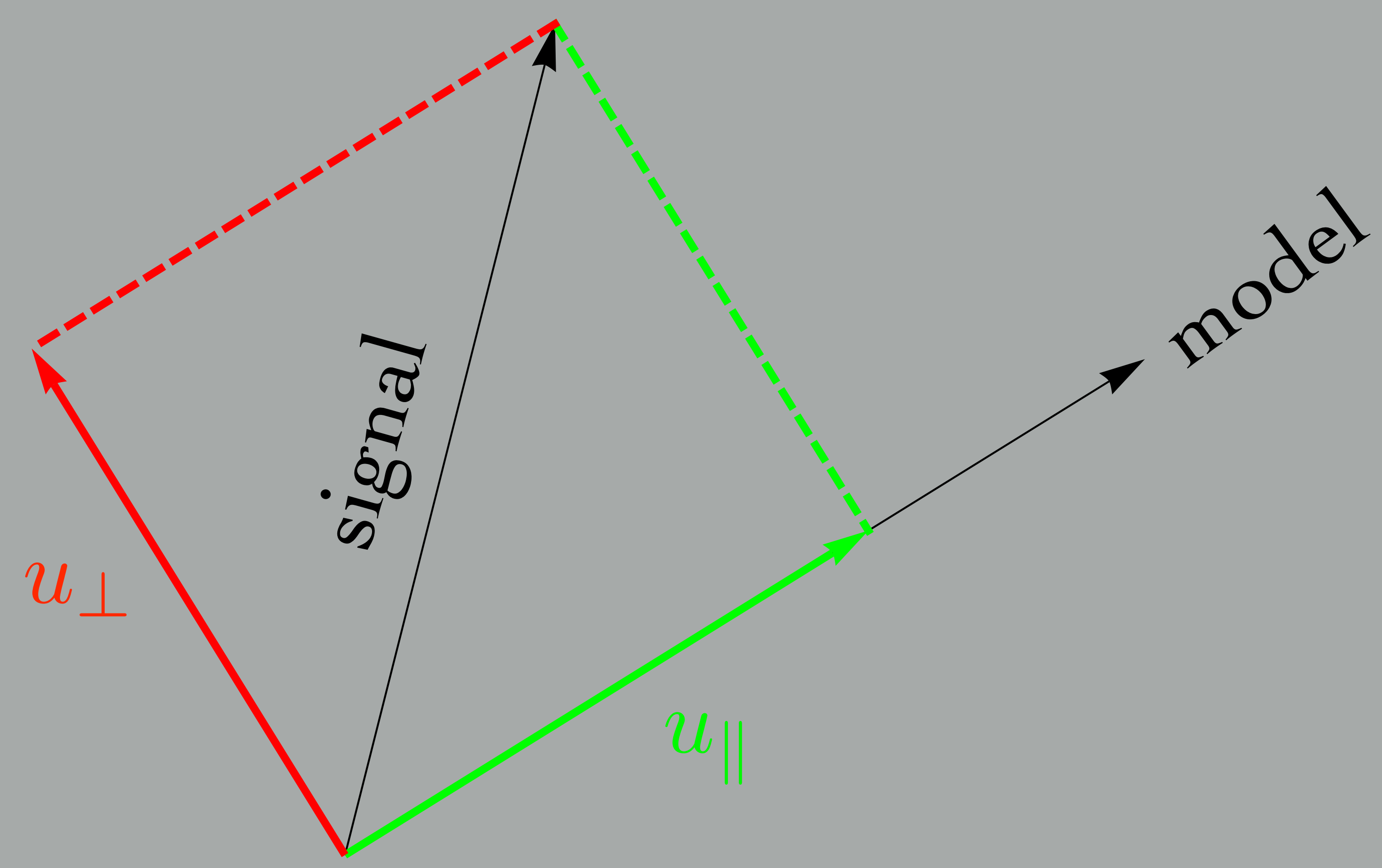


Projections



Projection

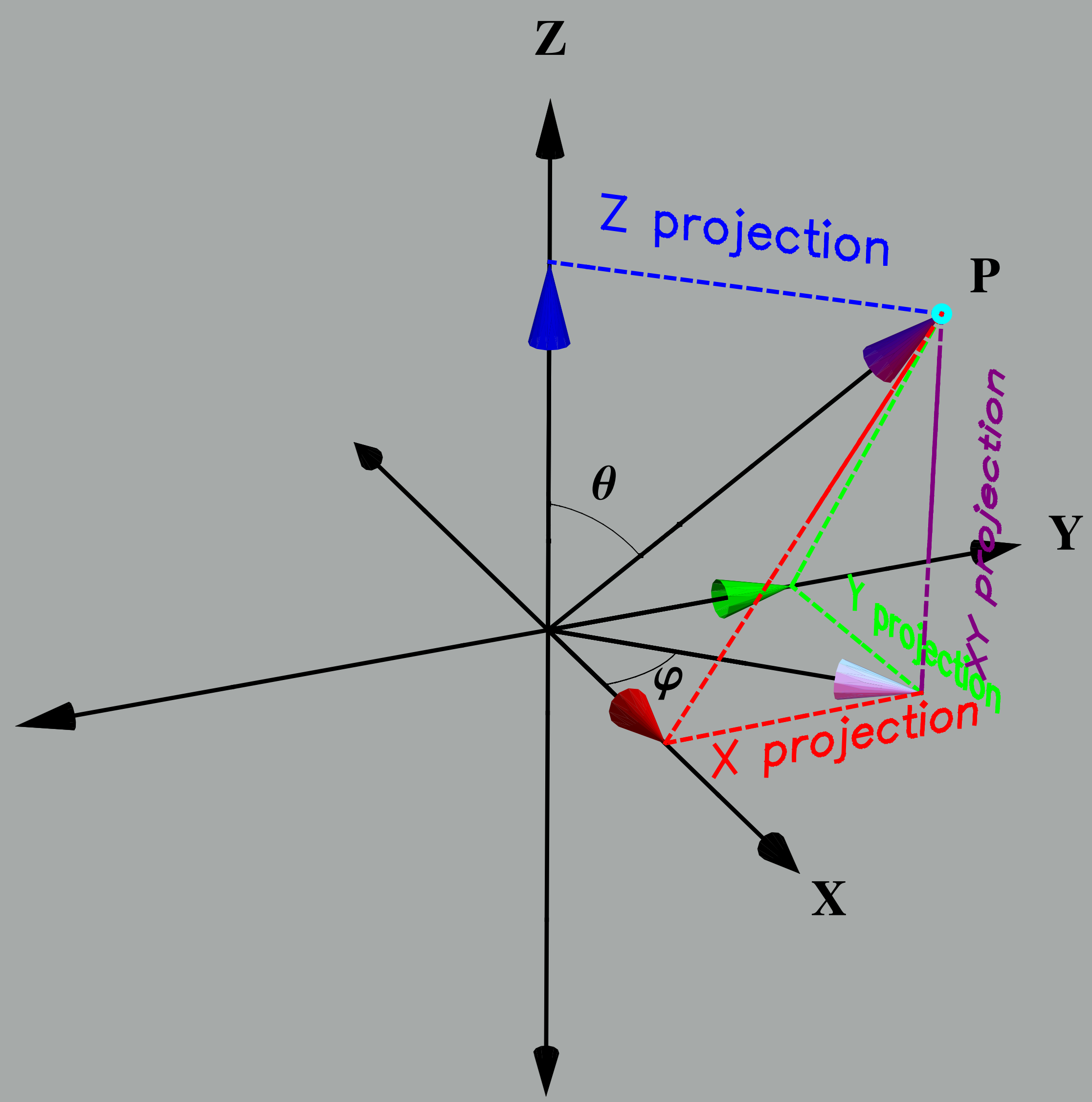
Example: Least squares fitting



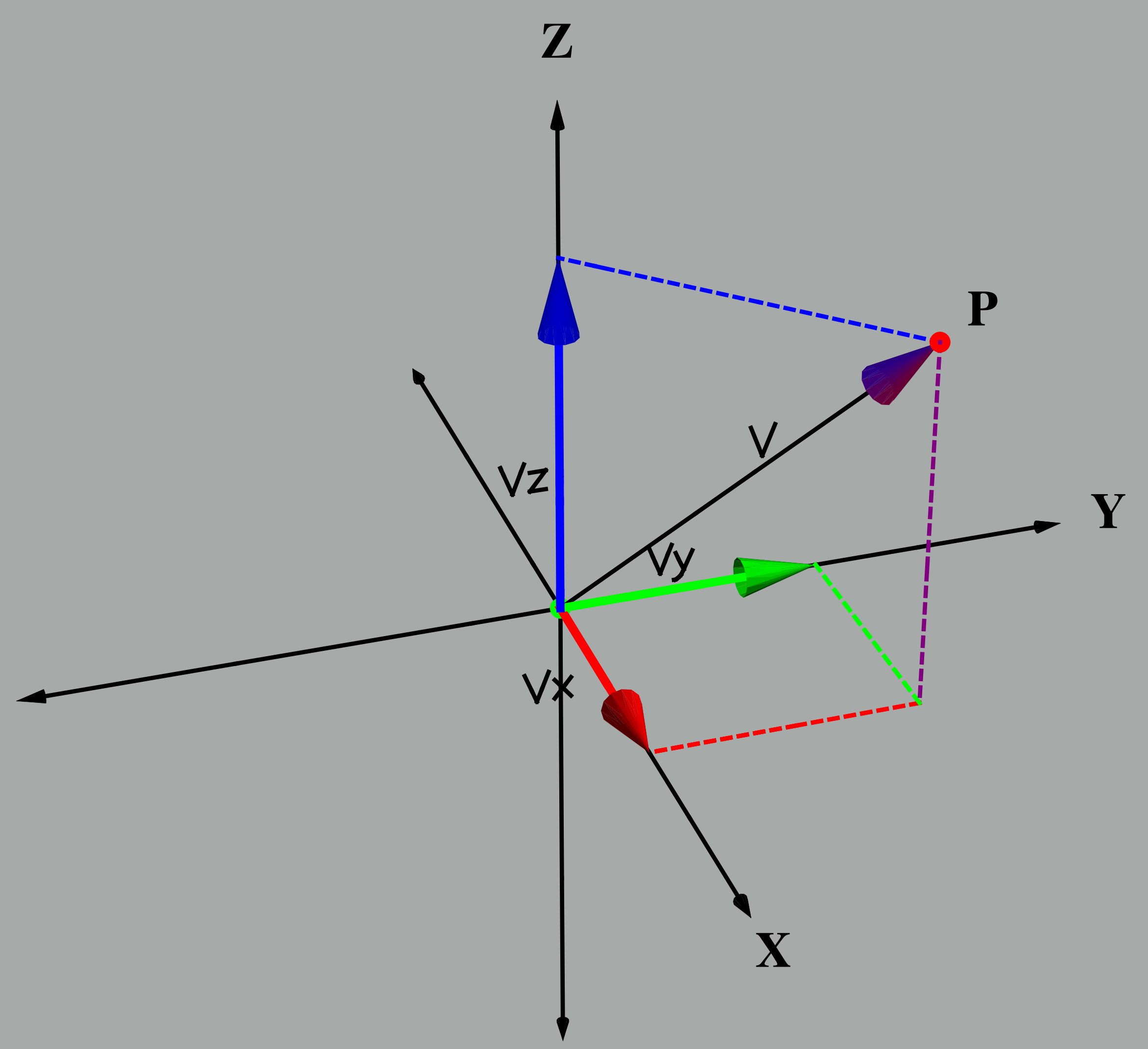
$u_\parallel$ = projection of signal onto model

$u_\perp$ = noise

Projections



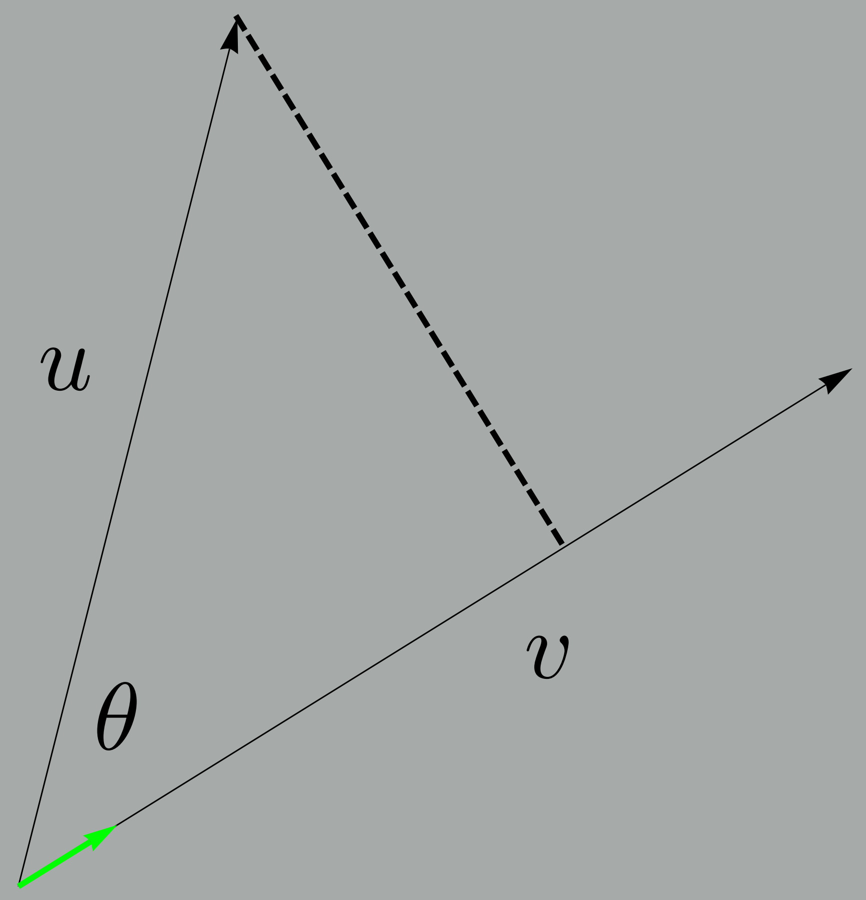
Vector components



In an *orthogonal* coordinate system, the vector length is the square root of the sum of the squares of the projections along the different axes

$$\|\boldsymbol{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\boldsymbol{v}^t \boldsymbol{v}}$$

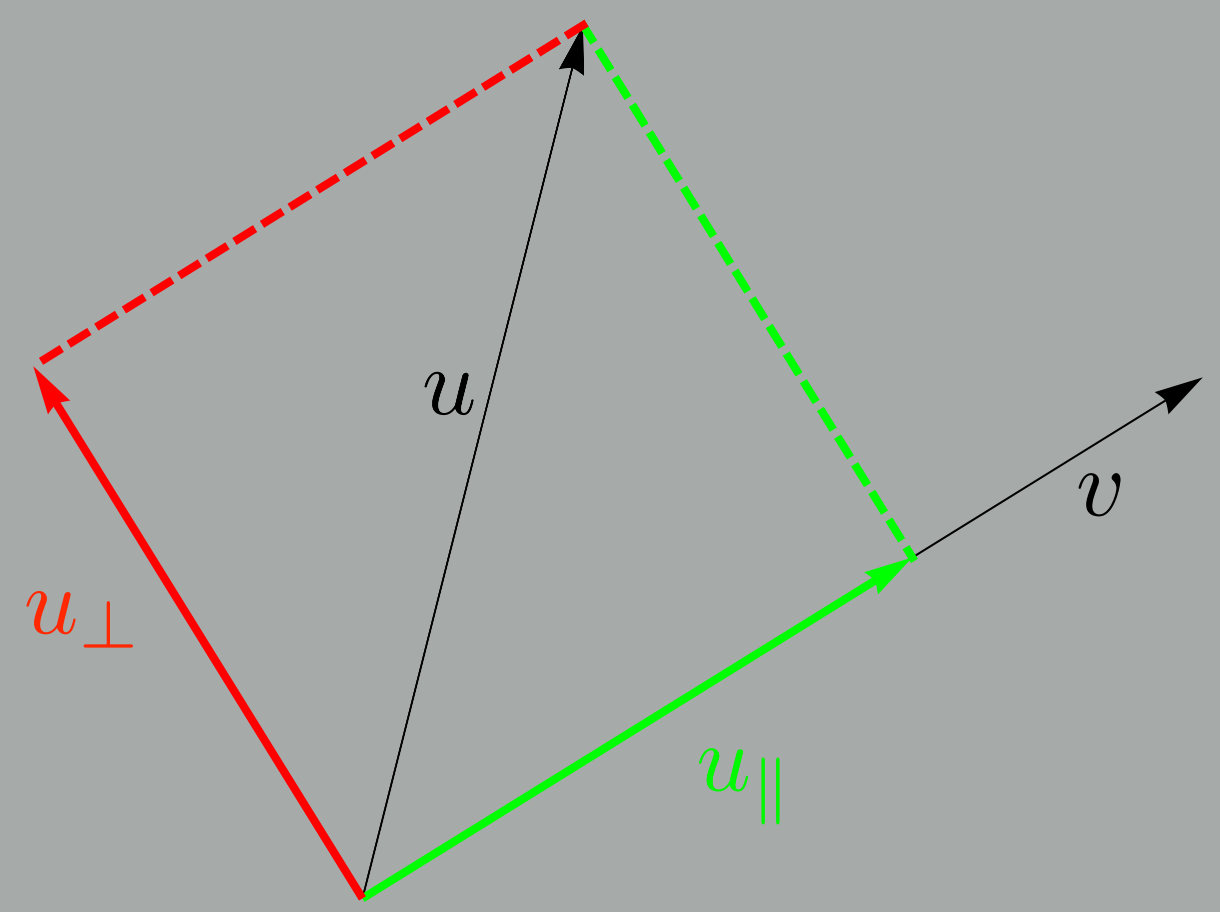
The Dot Product



$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos\theta$$

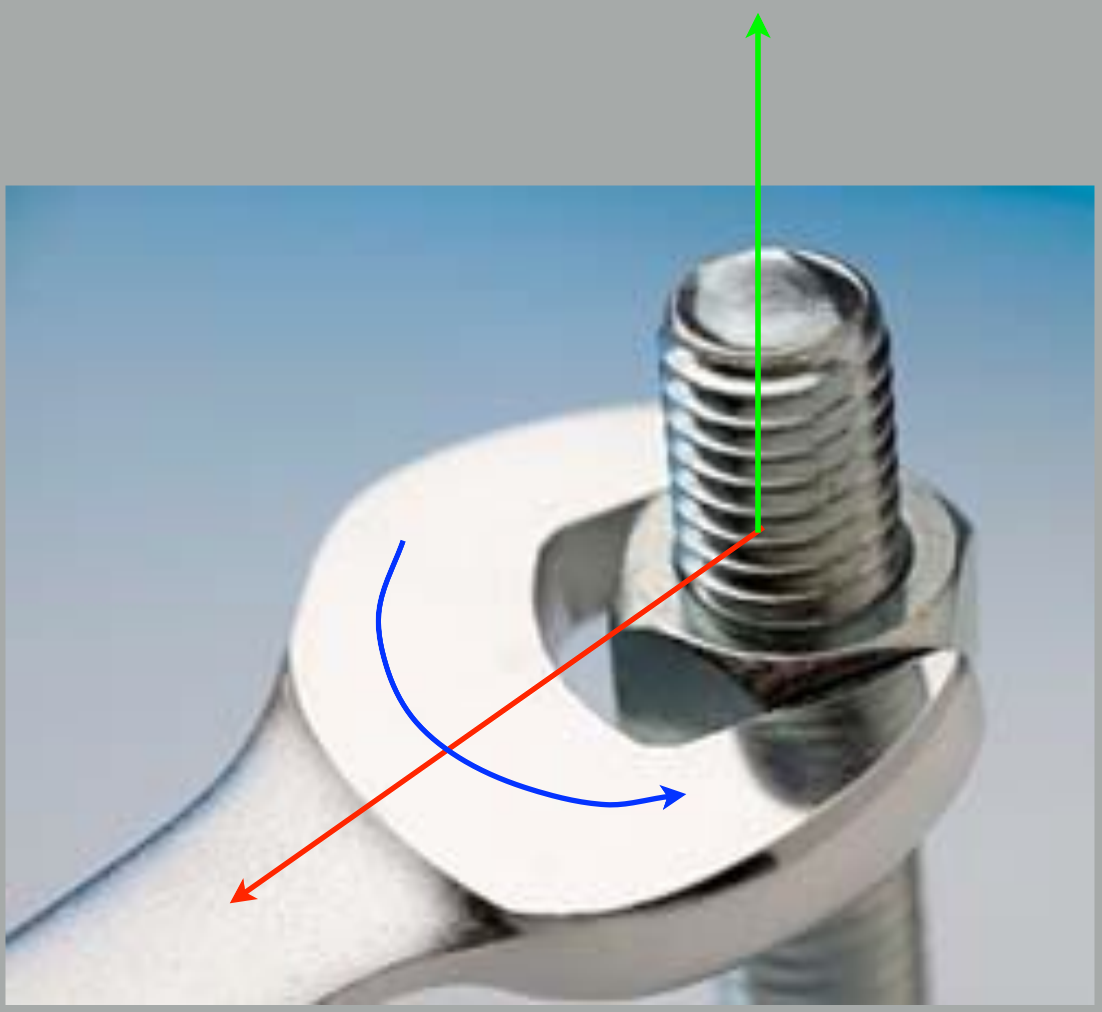
Projection

orthogonal complement: $\quad u_\perp = u - u_\parallel$



projection of u onto v: $\quad u_\parallel = \left(\frac{u \cdot v}{v \cdot v}\right) v$

The Cross Product

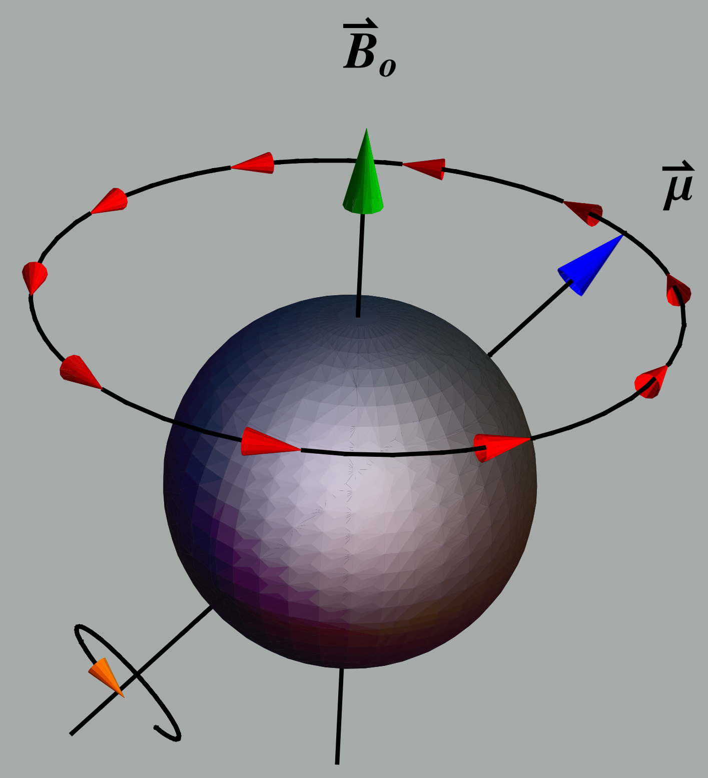


torque: $\tau = r \times F$

The Cross Product

angular momentum: $L = r \times p$

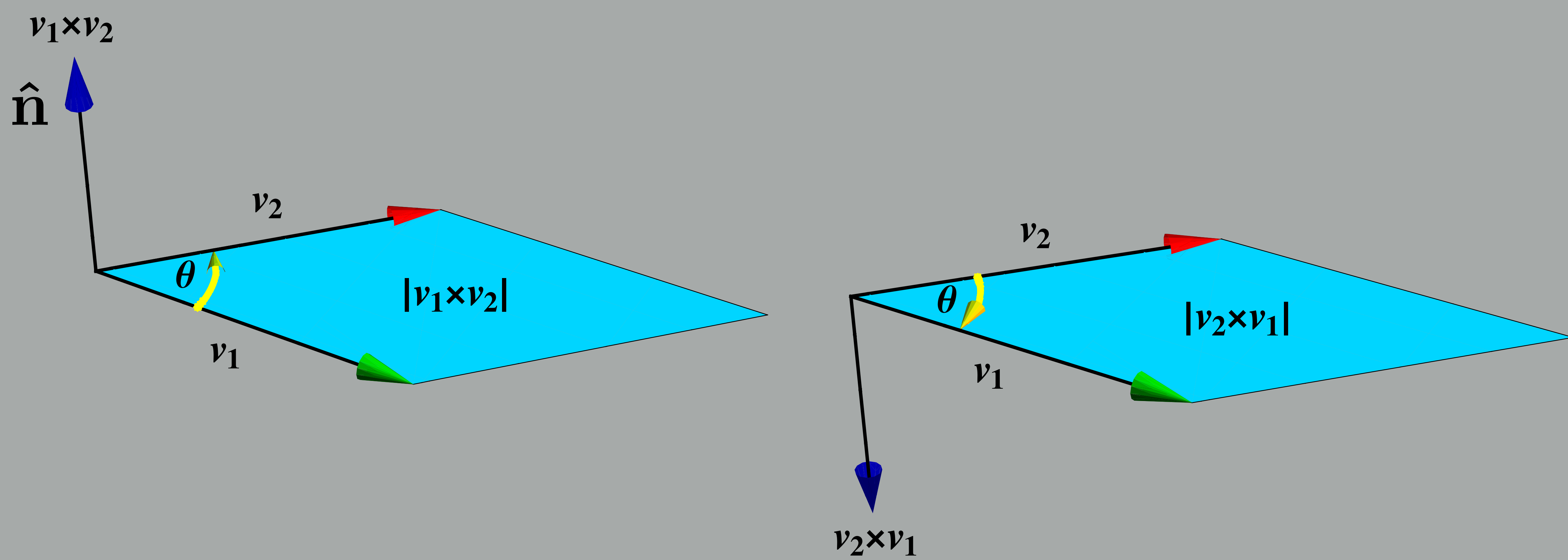
The Cross Product



nuclear precession

torque: $\tau = \mu \times B_0$

The Cross Product



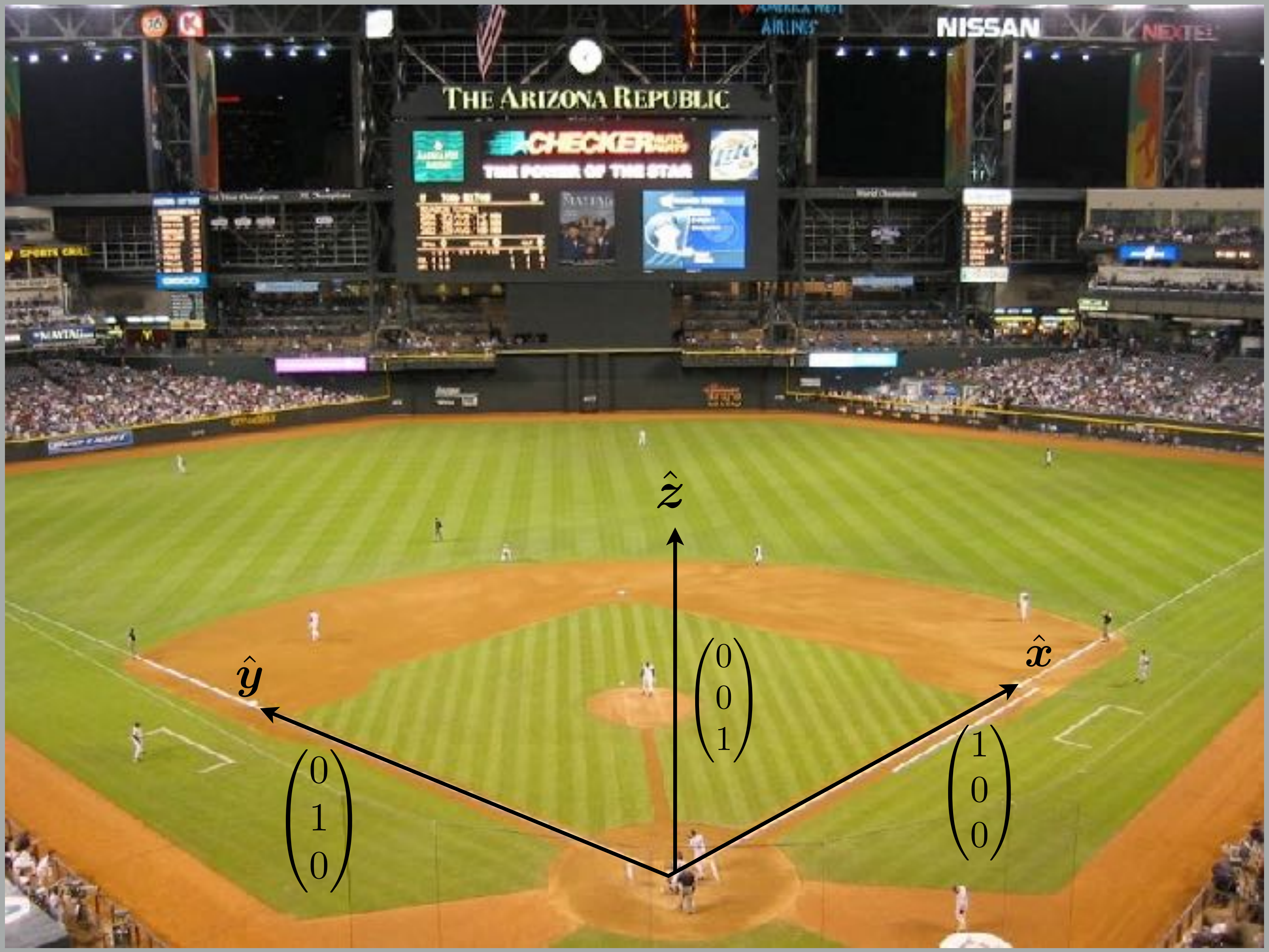
$$\mathbf{v_1} \times \mathbf{v_2} = |\mathbf{v_1}||\mathbf{v_1}|\sin\theta\hat{\mathbf{n}}$$

Matrices

Matrices as collections of vectors

Coordinate, in units of the distance to bases (90')

Coordinate system



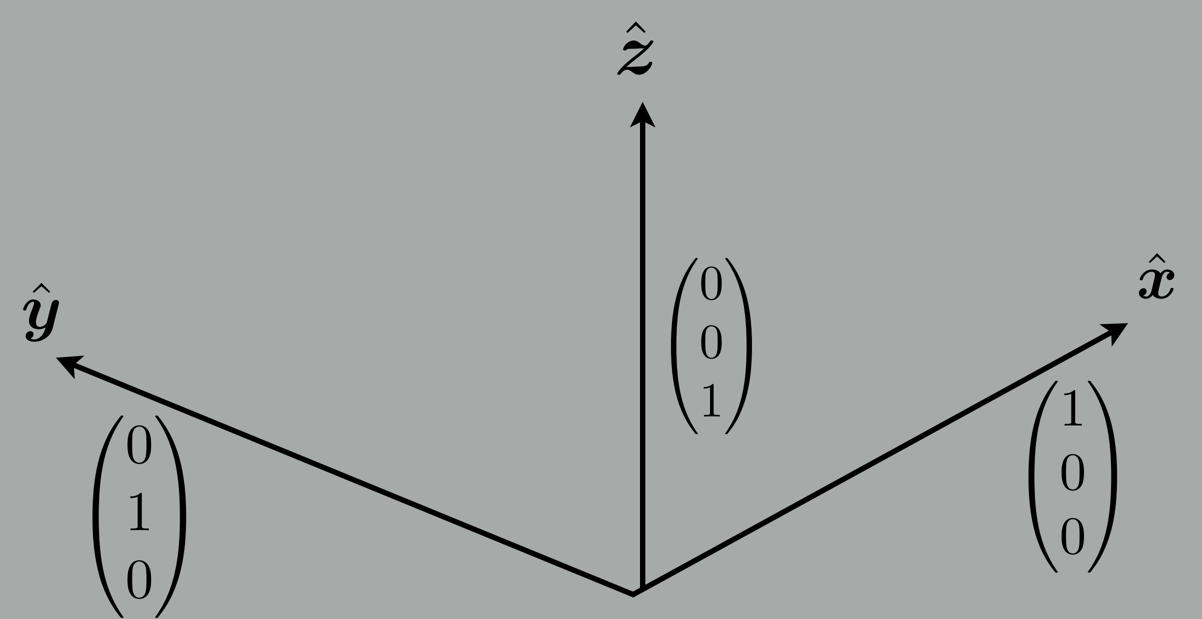
$$\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Coordinate, in units of the distance to bases (90')

Coordinate system

$$(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

"diagonal matrix"



Two-dimensional matrix

y

x

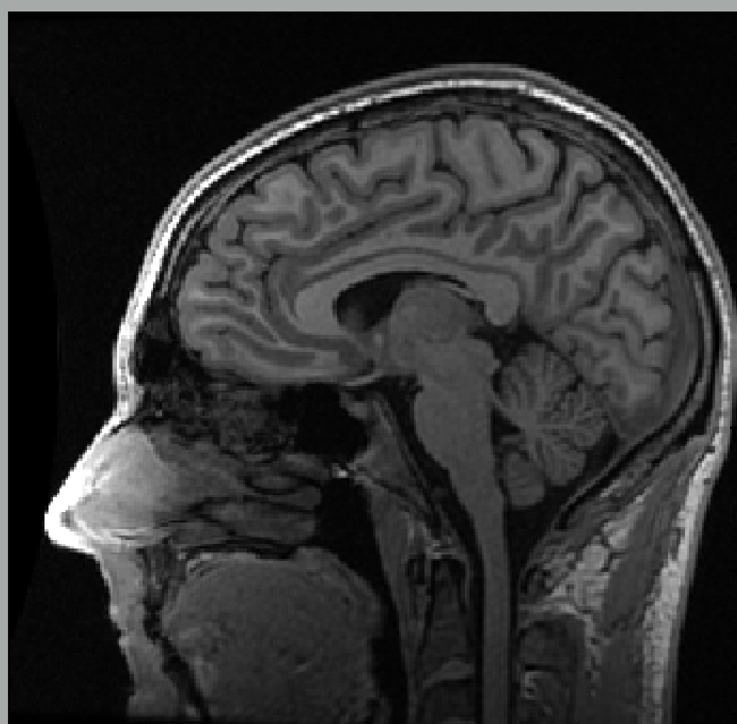
intensities as a function of x and y

Matrix Notation

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} =$$ 

$$a_{ij} = \text{intensity of voxel in row } i \text{ and column } j$$

Matrix addition

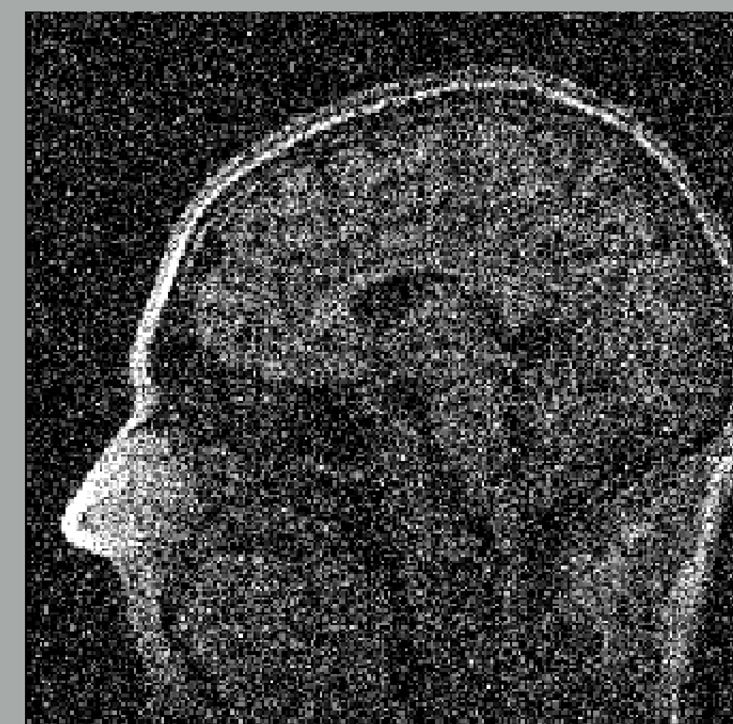


$\boldsymbol{A}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\boldsymbol{B}$

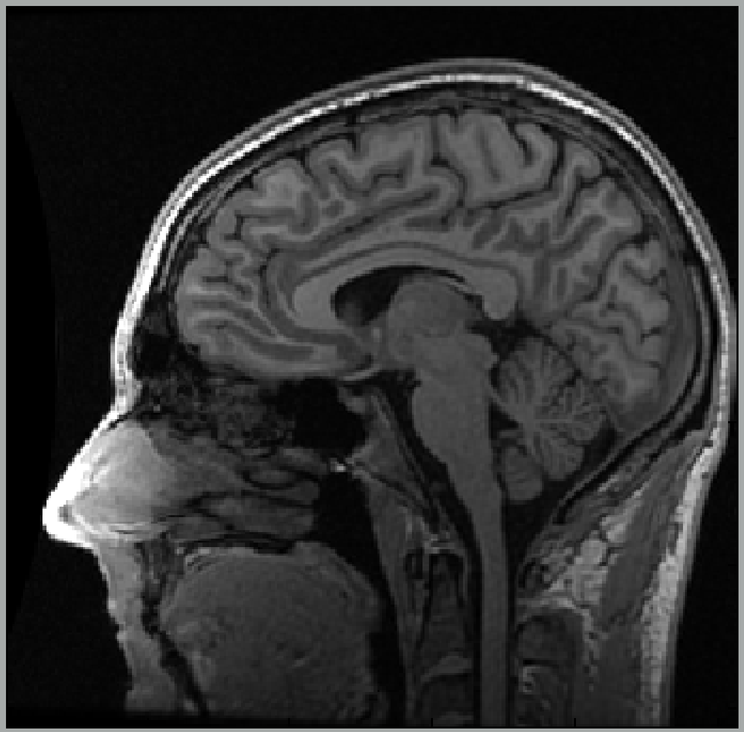
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$



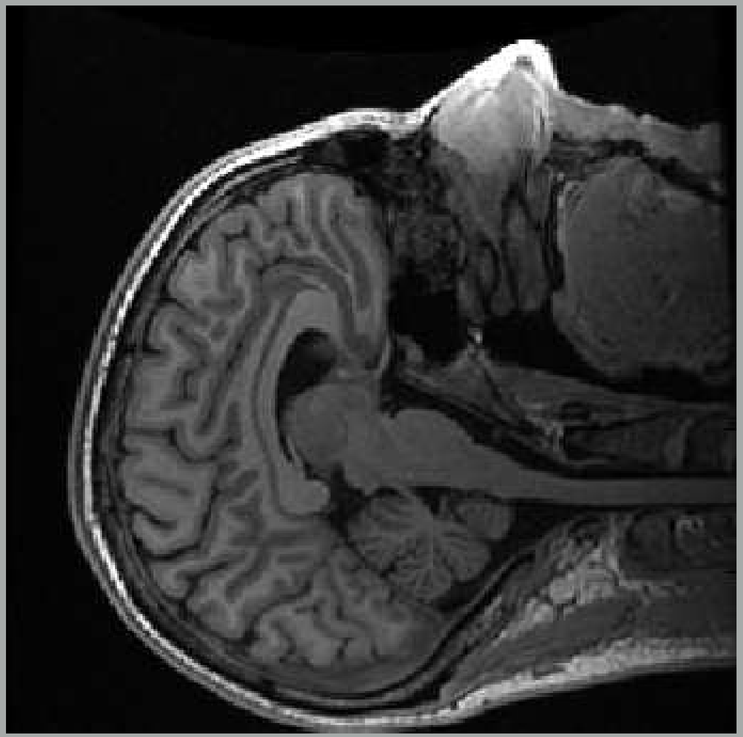
$\boldsymbol{A} + \boldsymbol{B}$

$$\begin{pmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{pmatrix}$$

Matrix transpose



$\boldsymbol{A}$

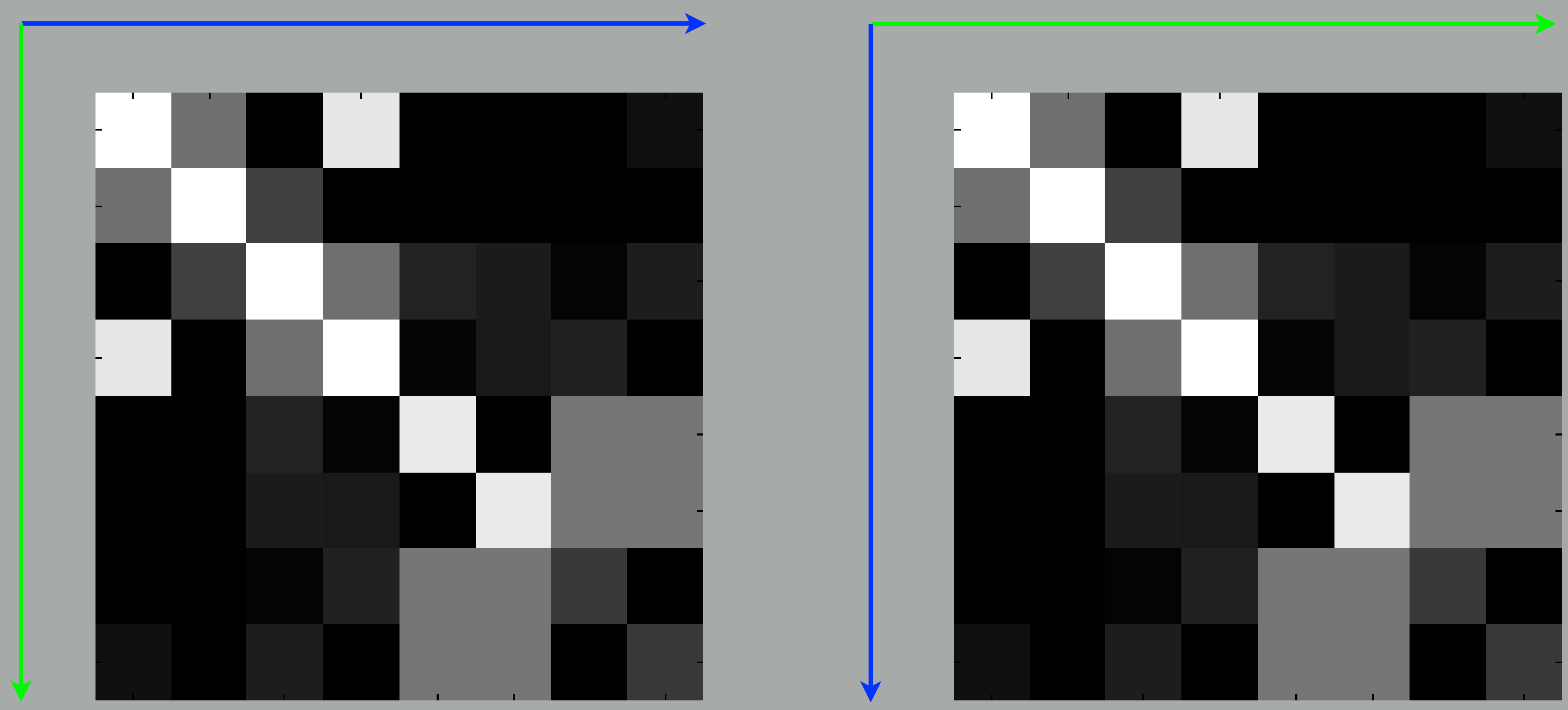


$\boldsymbol{A}^t$

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$\boldsymbol{A}^t = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

Symmetric Matrix



$$\boldsymbol{A} \quad = \quad \boldsymbol{A}^t$$

Example: Diffusion Tensor

$$\begin{pmatrix} D_{xx} & \boxed{D_{xy}} & \boxed{D_{xz}} \\ \boxed{D_{yx}} & D_{yy} & \boxed{D_{yz}} \\ \boxed{D_{zx}} & \boxed{D_{zy}} & D_{zz} \end{pmatrix} = \begin{pmatrix} D_{xx} & \boxed{D_{xy}} & \boxed{D_{xz}} \\ \boxed{D_{xy}} & D_{yy} & \boxed{D_{yz}} \\ \boxed{D_{xz}} & \boxed{D_{yz}} & D_{zz} \end{pmatrix}$$

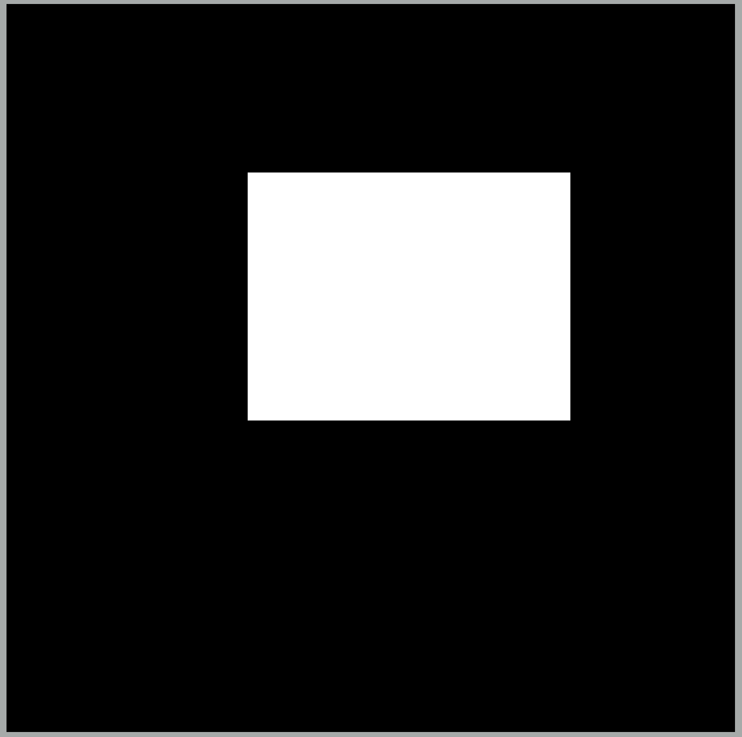
$$\boldsymbol{D} = \boldsymbol{D}^t$$

Matrix Multiplication

Hadamard product
Kroneker product
Dot product

Matrix Multiplication

Hadamard product



$\boldsymbol{A}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\boldsymbol{B}$

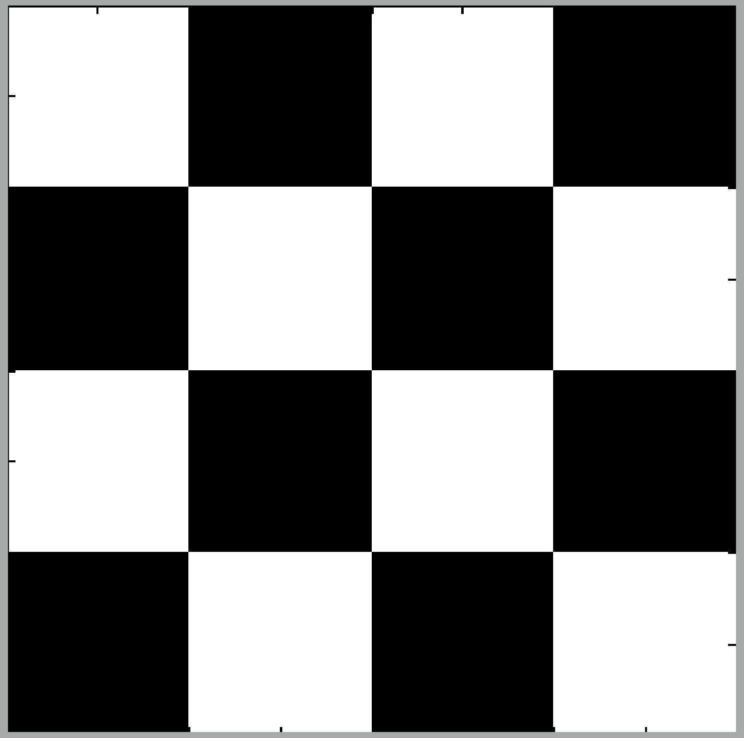
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$\boldsymbol{AB}$

$$\begin{pmatrix} \alpha a & \beta b \\ \gamma c & \delta d \end{pmatrix}$$

Matrix Multiplication

Kroneker product

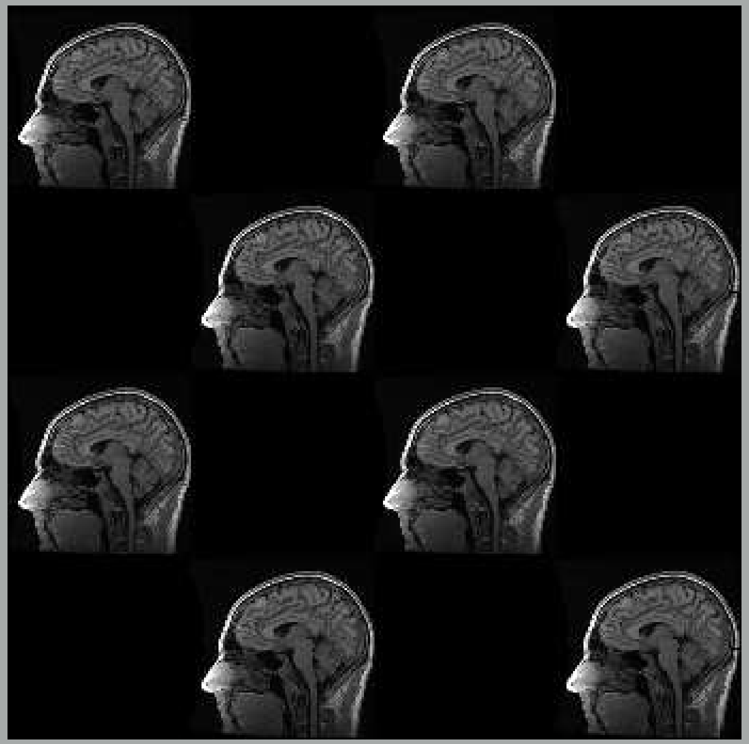


$\boldsymbol{A}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\boldsymbol{B}$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$



$\boldsymbol{A} \otimes \boldsymbol{B}$

$$\begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix}$$

Vector dot product

linear equation

$$f = \quad ax + by + cz$$

Write in vector form:

$$\boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$f = \underbrace{\begin{pmatrix} x & y & z \end{pmatrix}}_{\mathbf{u}^t} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{v}} = \mathbf{u}^t \mathbf{v}$$

Vector dot product

linear equation

$$f = \quad ax + by + cz$$

Write in vector form:

$$\boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$f = \underbrace{\begin{pmatrix} x & y & z \end{pmatrix}}_{\mathbf{u}^t} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{v}} = \mathbf{u}^t \mathbf{v}$$

Vector dot product

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Vector dot product

linear equation

$$f = \quad ax + by + cz$$

Write in vector form:

$$\boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$f = \underbrace{\begin{pmatrix} x & y & z \end{pmatrix}}_{\mathbf{u}^t} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{v}} = \mathbf{u}^t\mathbf{v}$$

Matrices dot product

a system of linear equations

$$\begin{aligned} f_1 &= a_1x + b_1y + c_1z \\ f_2 &= a_2x + b_2y + c_2z \\ f_3 &= a_3x + b_3y + c_3z \end{aligned}$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Matrices dot product

a system of linear equations

$$
\begin{aligned}
f_1 &= a_1 x + b_1 y + c_1 z \\
f_2 &= a_2 x + b_2 y + c_2 z \\
f_3 &= a_3 x + b_3 y + c_3 z
\end{aligned}
$$

$$
\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
$$

Matrices dot product

a system of linear equations

$$\begin{aligned} f_1 &= a_1x + b_1y + c_1z \\ f_2 &= a_2x + b_2y + c_2z \\ f_3 &= a_3x + b_3y + c_3z \end{aligned}$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Identity Matrix

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{I}\boldsymbol{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{x}$$

Matrix dot product

What about

$$\boldsymbol{AB} = \underbrace{\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_3 \end{pmatrix}}_{B}$$

Matrices dot product

$$\begin{pmatrix} & & \\ & ? & \\ & & \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_3 \end{pmatrix}$$

Matrices dot product

$$\begin{pmatrix} \boxed{f_{11}} & & \\ & & \\ & & \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_3 \end{pmatrix}$$

Matrices dot product

$$\begin{pmatrix} f_{11} & f_{12} \\ & \\ & \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_3 \end{pmatrix}$$

Matrices dot product

$$
\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & \\ & \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_3 \end{pmatrix}
$$

Matrices dot product

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & \boxed{f_{22}} \\ & \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_3 \end{pmatrix}$$

Matrices dot product

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_3 \end{pmatrix}$$

Matrices dot product

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_3 \end{pmatrix}$$

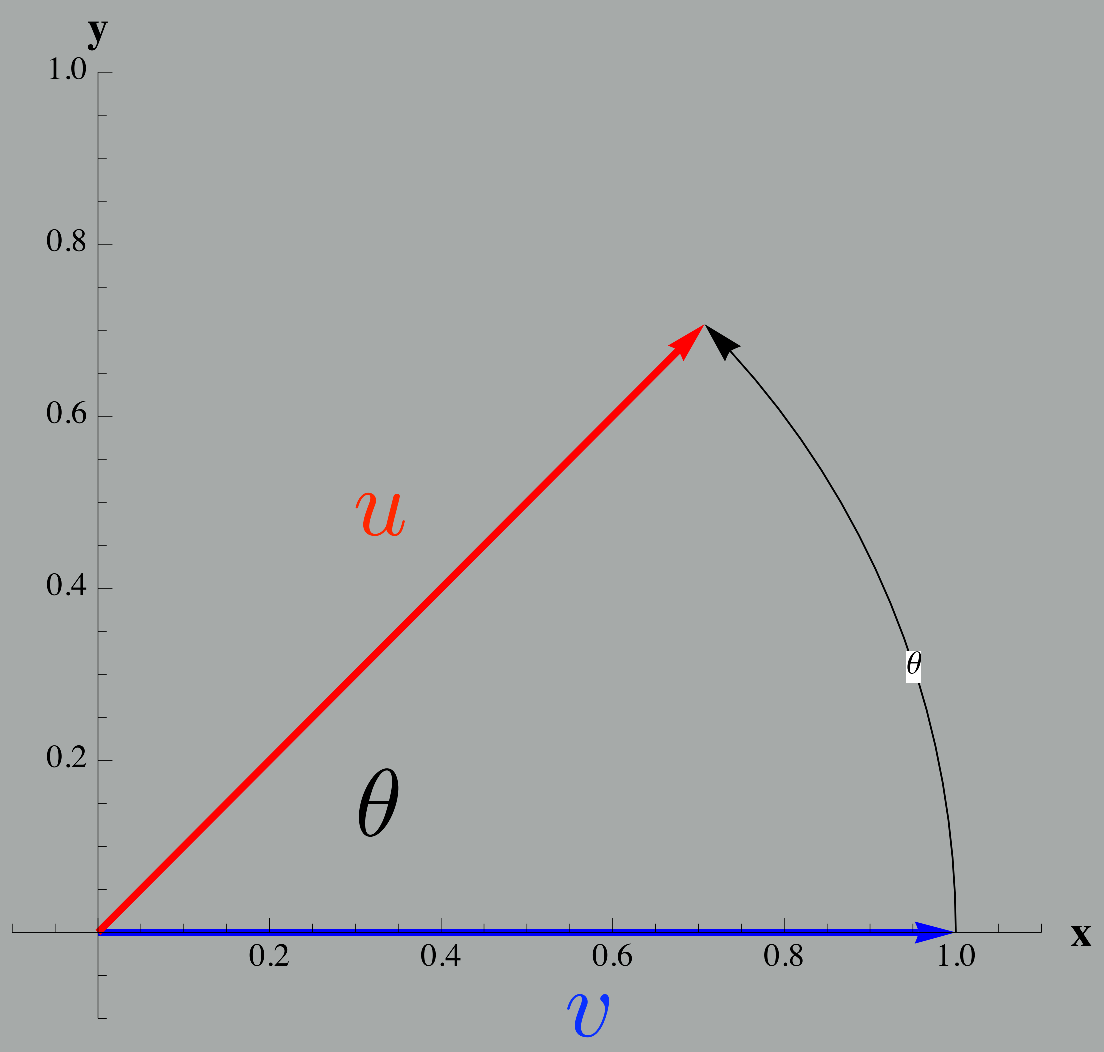
Matrices dot product

$$\text{dimensions} = [\text{rows} \times \text{cols}]$$

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_3 \end{pmatrix}$$

$$[n \times p] \qquad [n \times m] \qquad [m \times p]$$

Example: Rotations

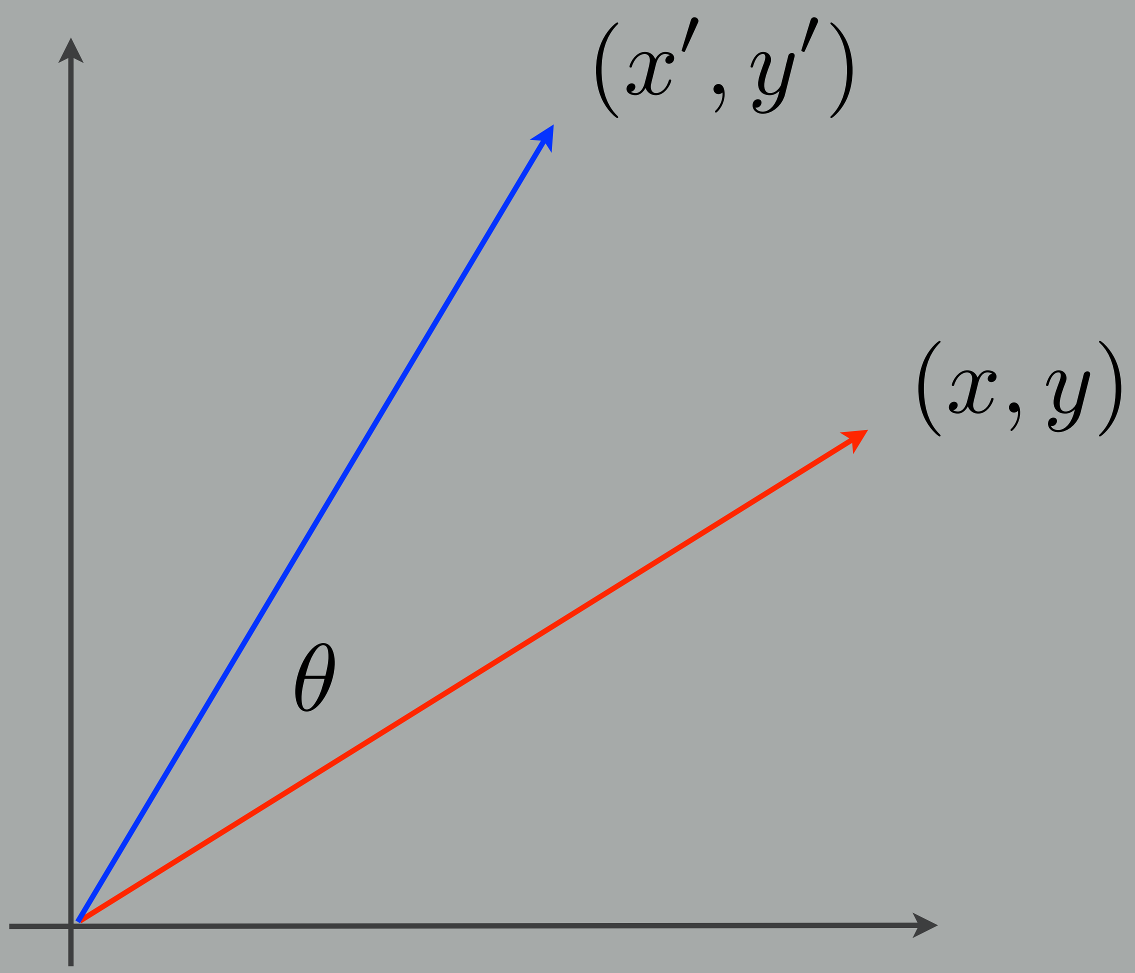


$R(\ \ ) =$

$$u = Rv$$

Rotations

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

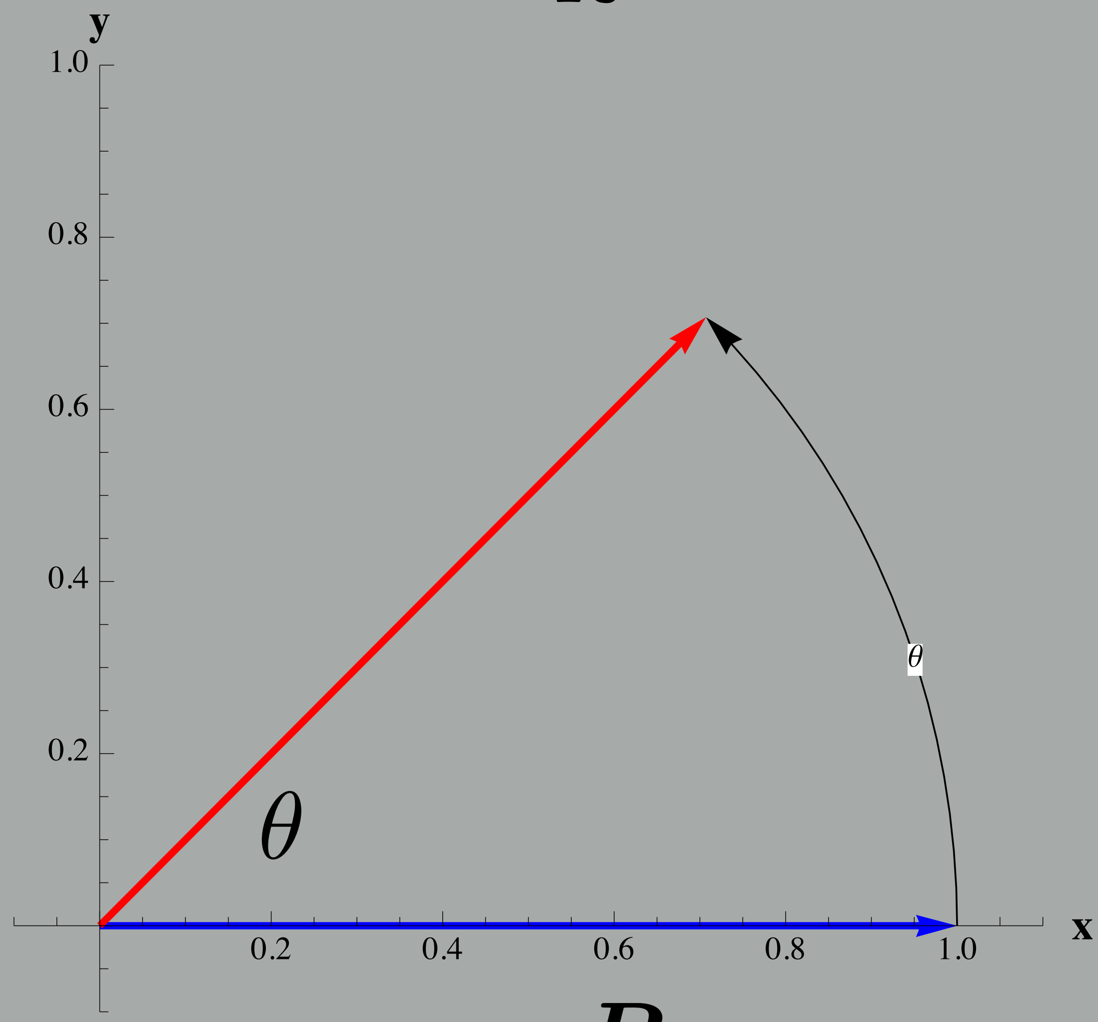


Rotations

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Rotations

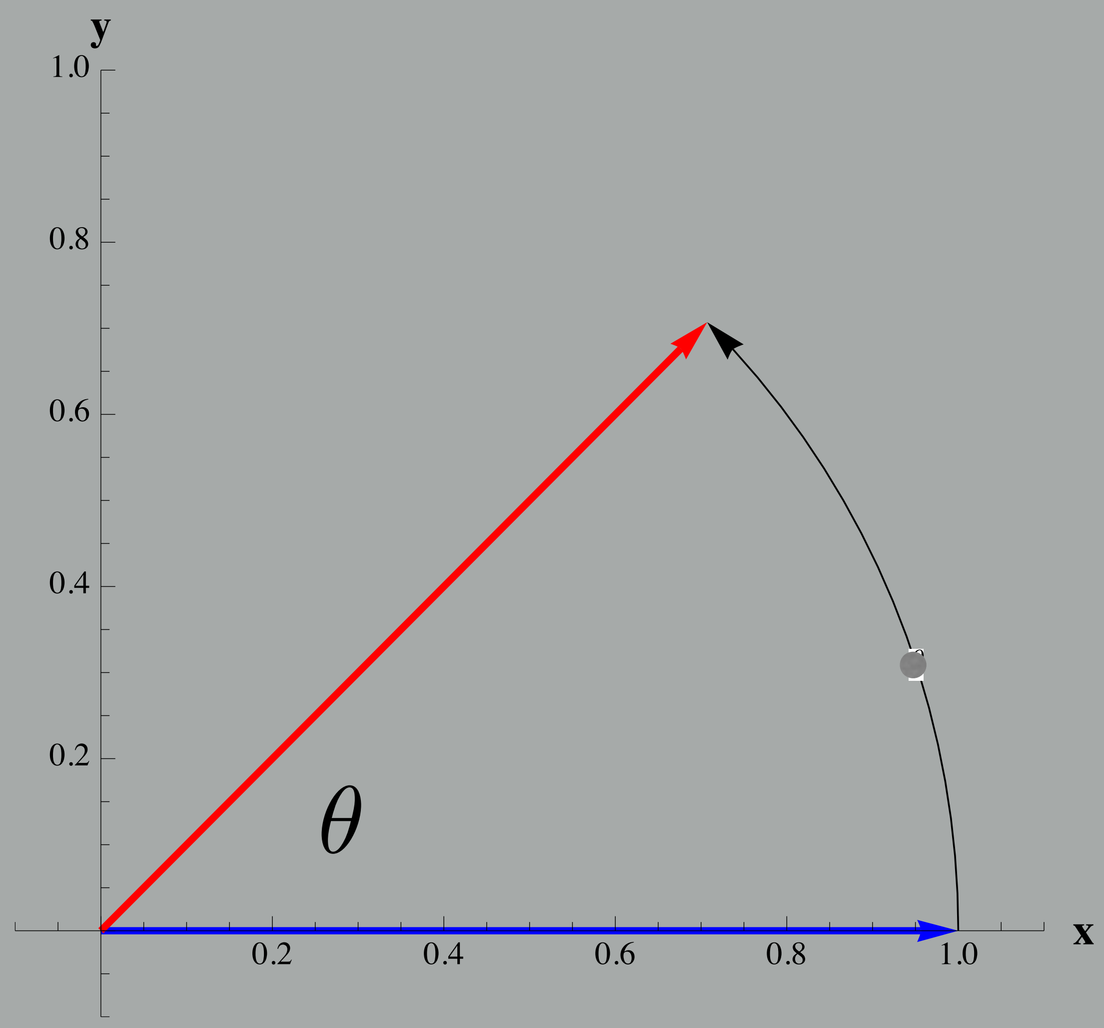
$$\underbrace{\begin{pmatrix} u_x \\ u_y \end{pmatrix}}_{\boldsymbol{u}} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\boldsymbol{R}} \underbrace{\begin{pmatrix} v_x \\ v_y \end{pmatrix}}_{\boldsymbol{v}}$$

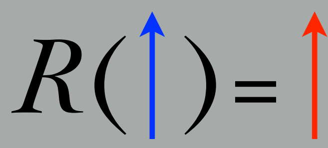


$$\boldsymbol{u} = \boldsymbol{R}\boldsymbol{v}$$

2D Rotation Matrix

$$\boldsymbol{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$





$$\boldsymbol{u} = \boldsymbol{R}\boldsymbol{v}$$

What about rotating an ellipse?

Equation of ellipse

$$Ax^2 + 2Bxy + Cy^2$$

In matrix form

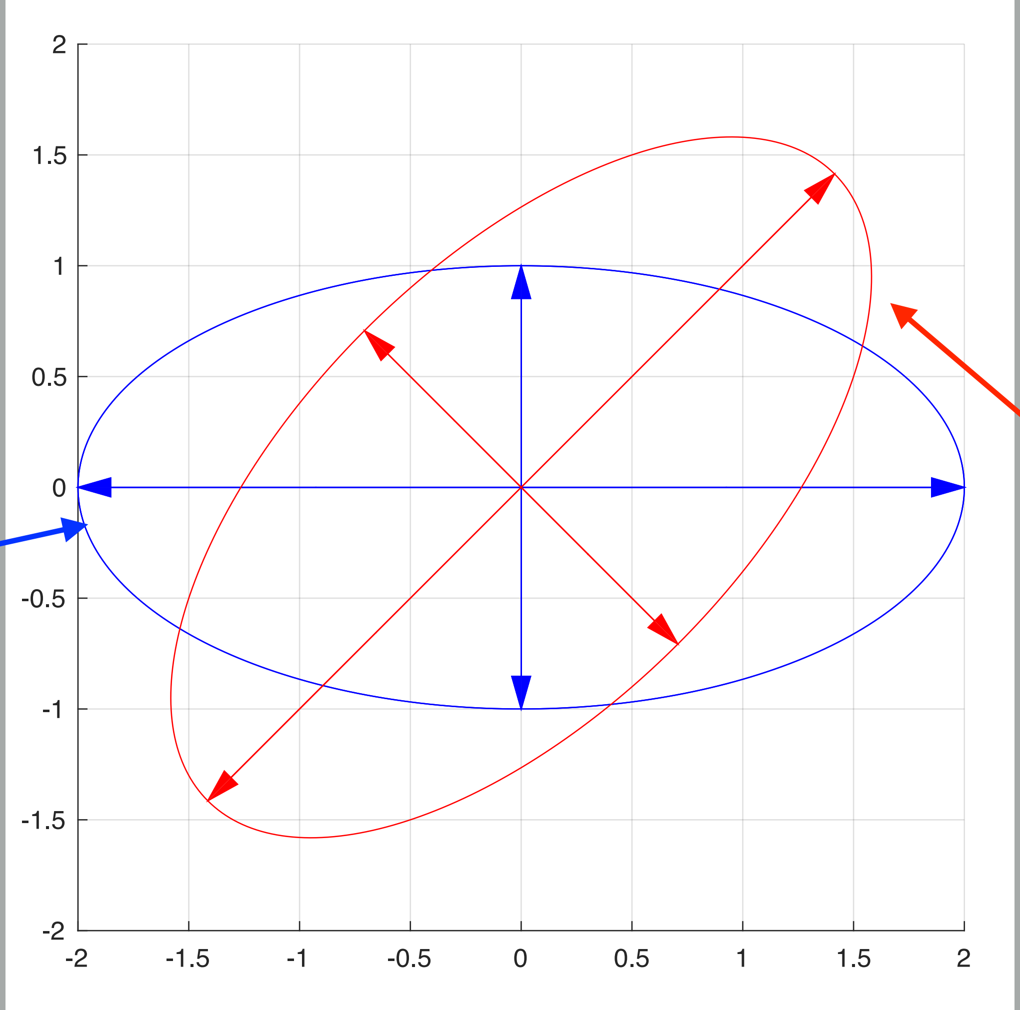
$$\vec{x}^t \vec{Q} \vec{x}$$

where $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\vec{Q} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$

What about rotating an ellipse?

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{x'}^t = \vec{R}\vec{x}$$



$$\vec{x}^t\vec{Q}\vec{x}$$

$$\vec{x'}^t\vec{Q}\vec{x'}$$

What about rotating an ellipse?

rotated axes: $x' = Rx$ ellipse along rotates axes: $x'^t Q x'$

$$x'^t Q x' = (Rx)^t Q (Rx) = x^t R^t Q R x = x^t Q' x$$

where $Q' \equiv R^t Q R$

Similarity transform

rotation of ellipse is different than rotation of vector!

Complex number motivation

$$\boxed{\text{MR image}}$$

$$m_\perp(\boldsymbol{x}) = \int s(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{k}$$

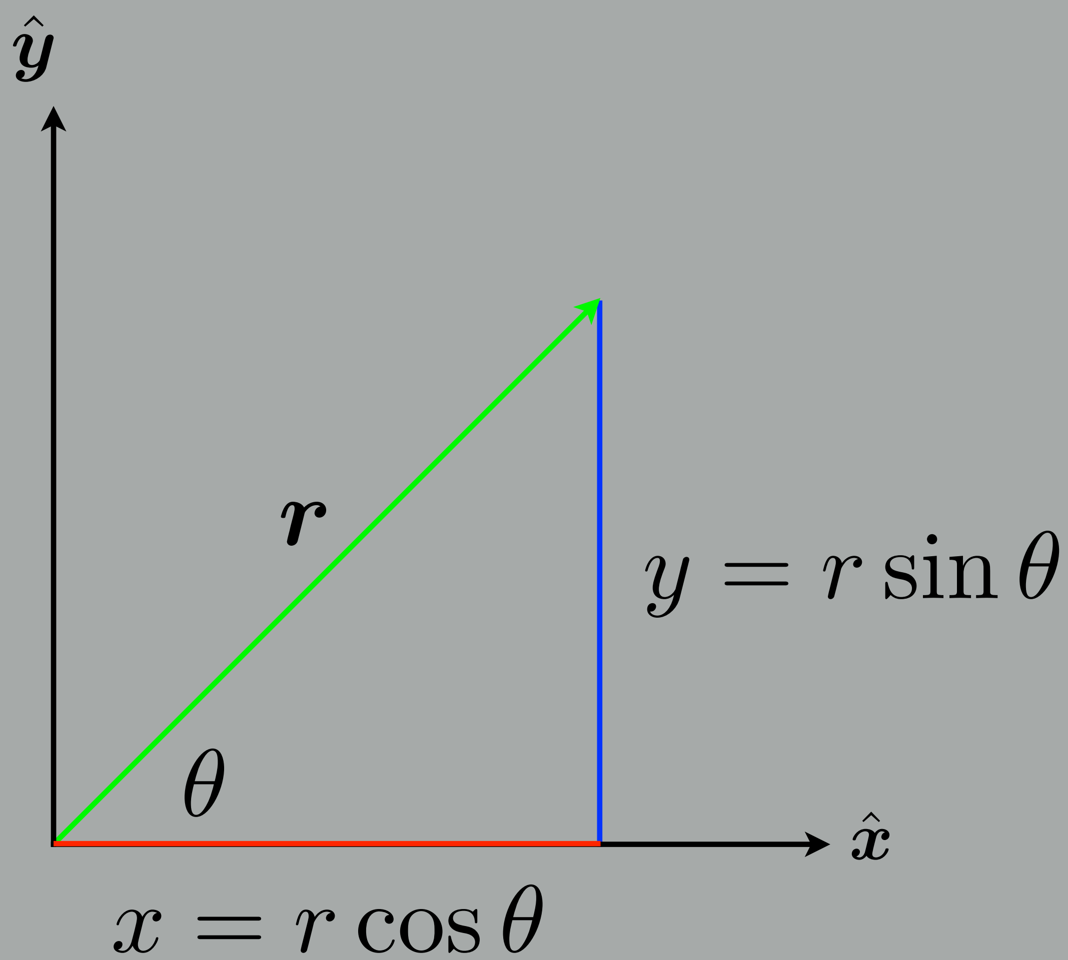
Euler's Relation

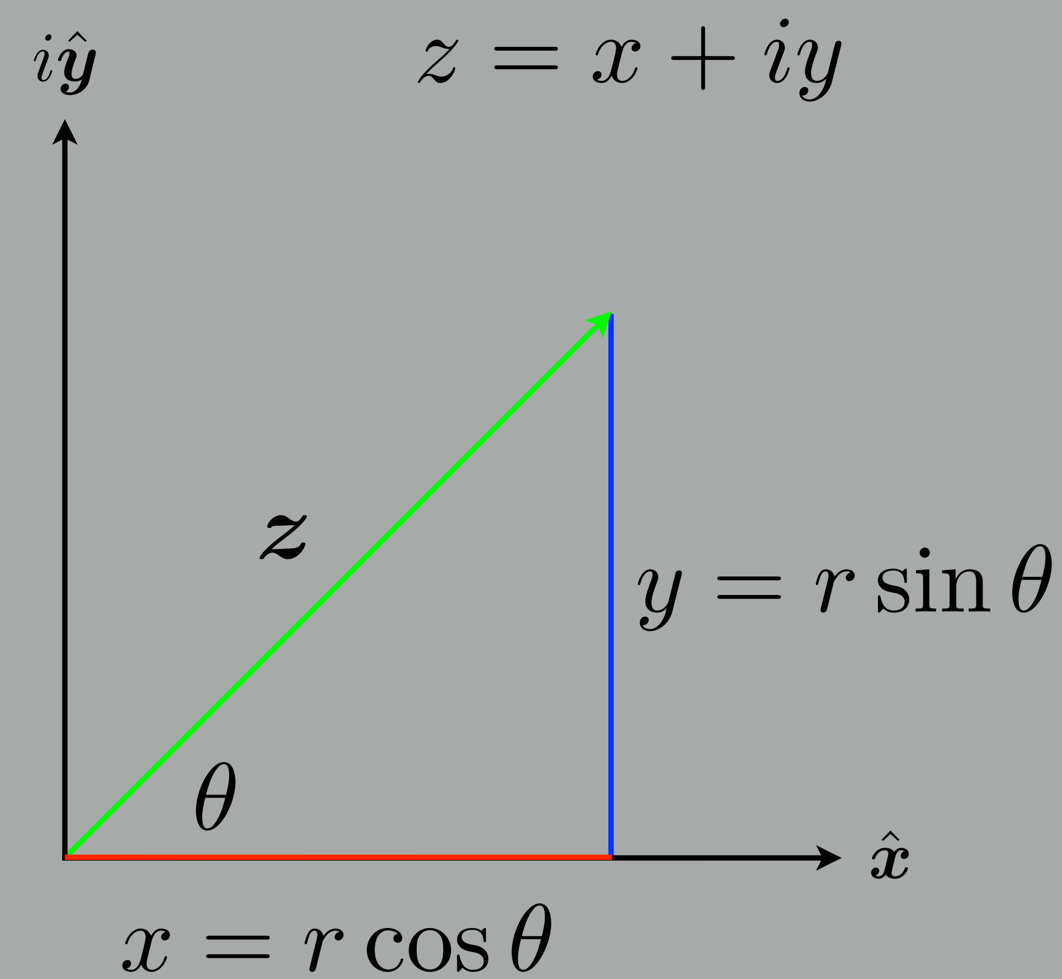
$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

This is the key to the relationship between *complex numbers* and *rotations*

$$\boxed{i = \sqrt{-1}}$$

The Complex Plane

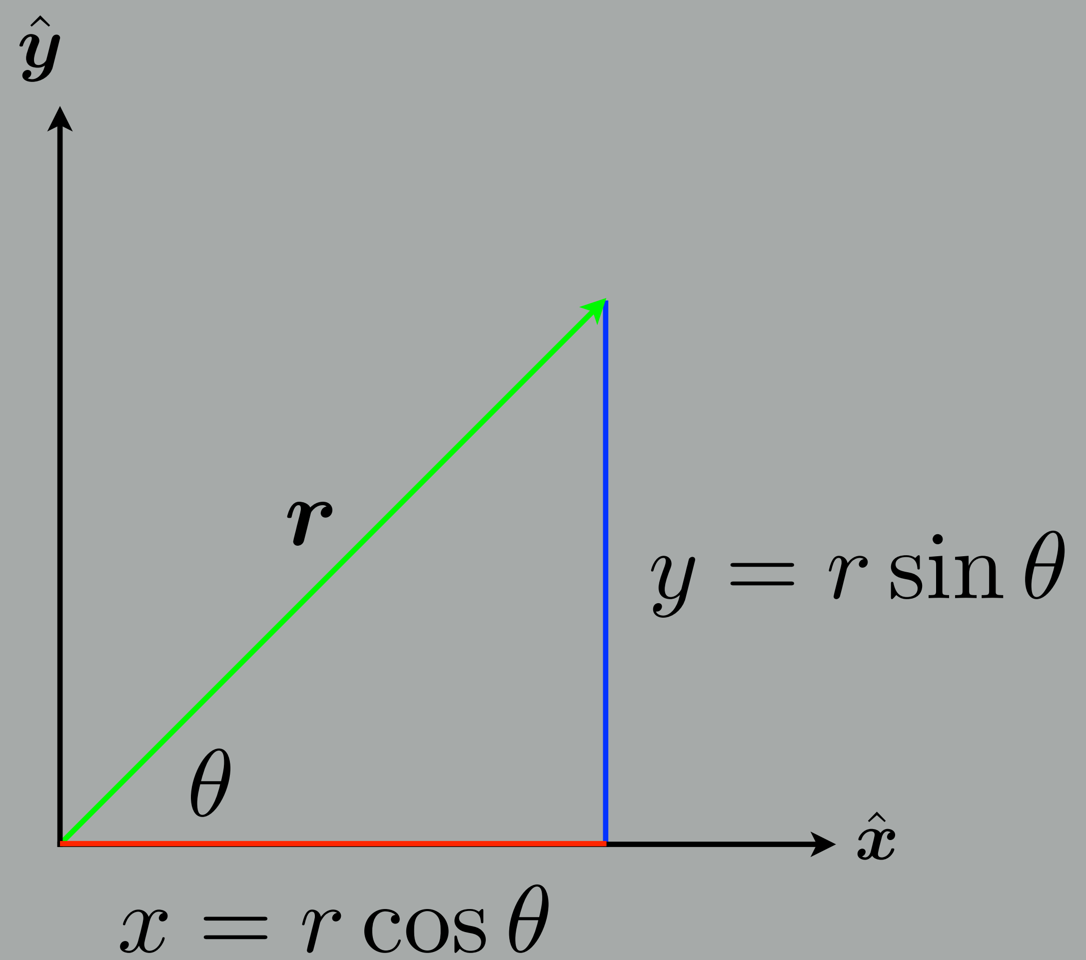


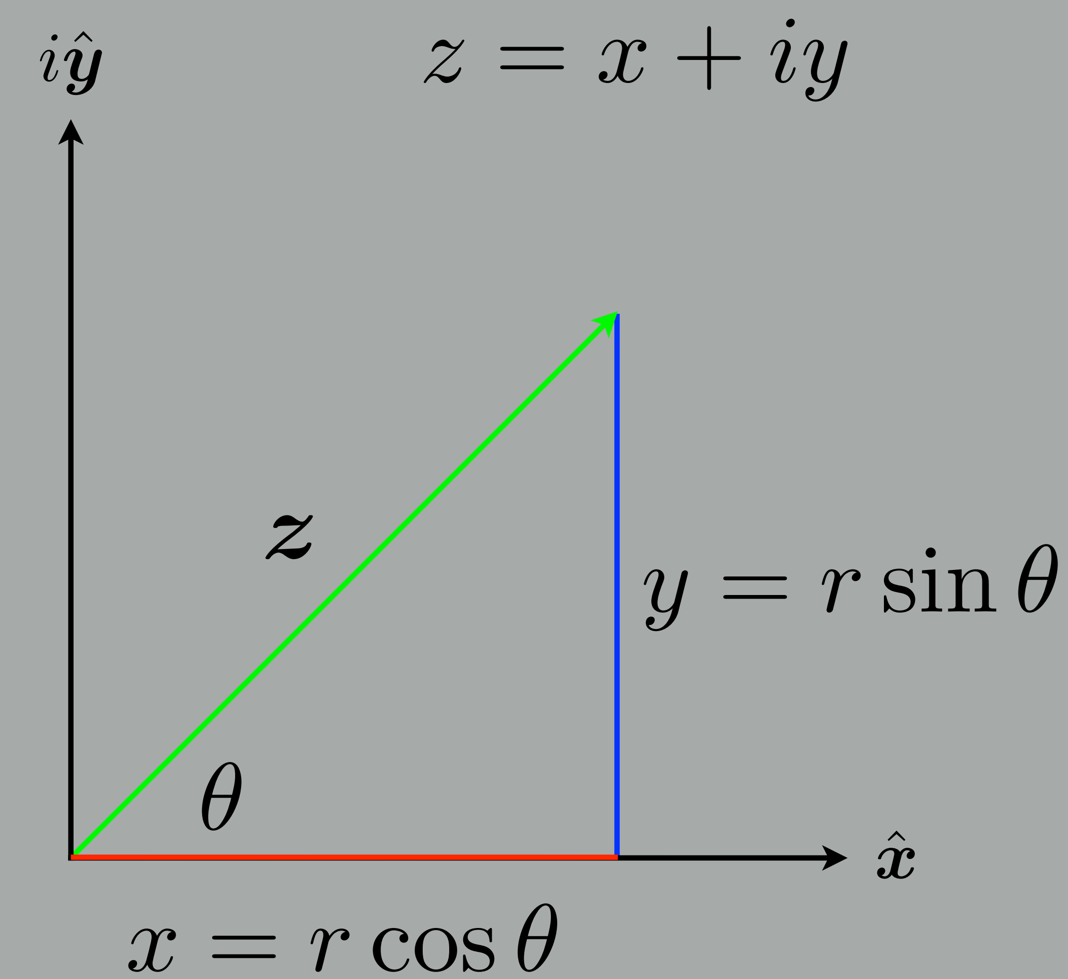


$$\|\boldsymbol{r}\| = \sqrt{x^2 + y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = \sqrt{r^2}$$

What is $\|\boldsymbol{z}\|$?

The Complex Plane





$$\|\boldsymbol{r}\| = \sqrt{x^2 + y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = \sqrt{r^2}$$

it has to be: $\|\boldsymbol{z}\| = r$

Imaginary Numbers

$$\boxed{i = \sqrt{-1}}$$

$$w = ia \ , \ a \text{ is real } (a \in \Re)$$

What is magnitude of w?

$$w^2 = i^2 a^2 = -a^2$$

Oops! Can't have negative magnitude

Complex Conjugate

$$w = \quad ia$$

$\boxed{a \text{ is real}, w \text{ is complex}}$

Define: $$w^* = -ia$$

$\boxed{\text{complex conjugate of } w}$

Replace i with -i

Notice what happens when we multiple
a complex number by its complex conjugate

$$ww^* = (ia)(-ia) = -i^2a^2 = a^2$$

we get a real number

Complex Number

$$z \;\; = x + iy \; , \;\; x \text{ and } y \text{ real}$$

$$z^* = x - iy$$ *Complex conjugate*: Replace i with $-i$

Magnitude of z

$$zz^* = (x + iy)(x - iy) = x^2 + y^2$$

$$\|\boldsymbol{z}\| = \sqrt{\boldsymbol{z}^* \boldsymbol{z}}$$

Complex Conjugation

Now let's create a complex number using Euler's relation multiplied by a real number *a*

$$w \;\; = ae^{i\theta} \;\; = a(\cos\theta + i\sin\theta)$$

and create its complex conjugate:

$$w^* = ae^{-i\theta} = a(\cos\theta - i\sin\theta)$$

and confirm the result from the previous page:

$$ww^* = (ae^{i\theta})(ae^{-i\theta}) = a^2 e^{i\theta - i\theta} = a^2 e^0 = a^2$$

Complex Number

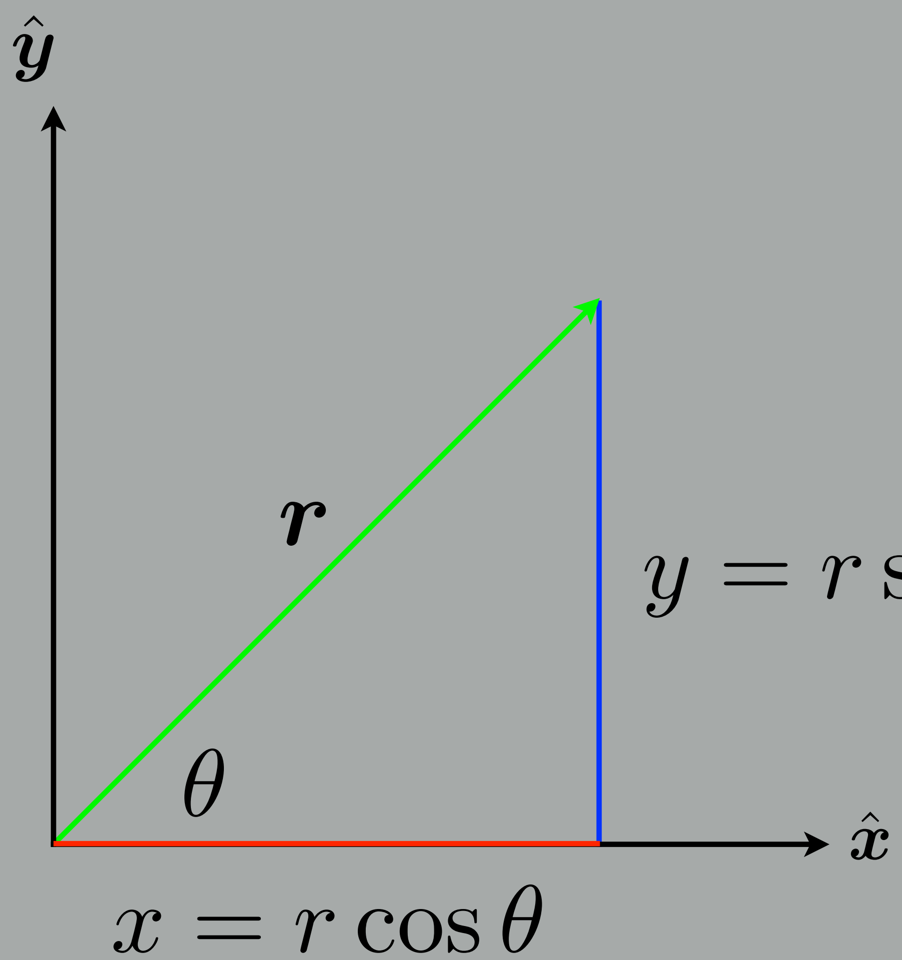
Magnitude of z

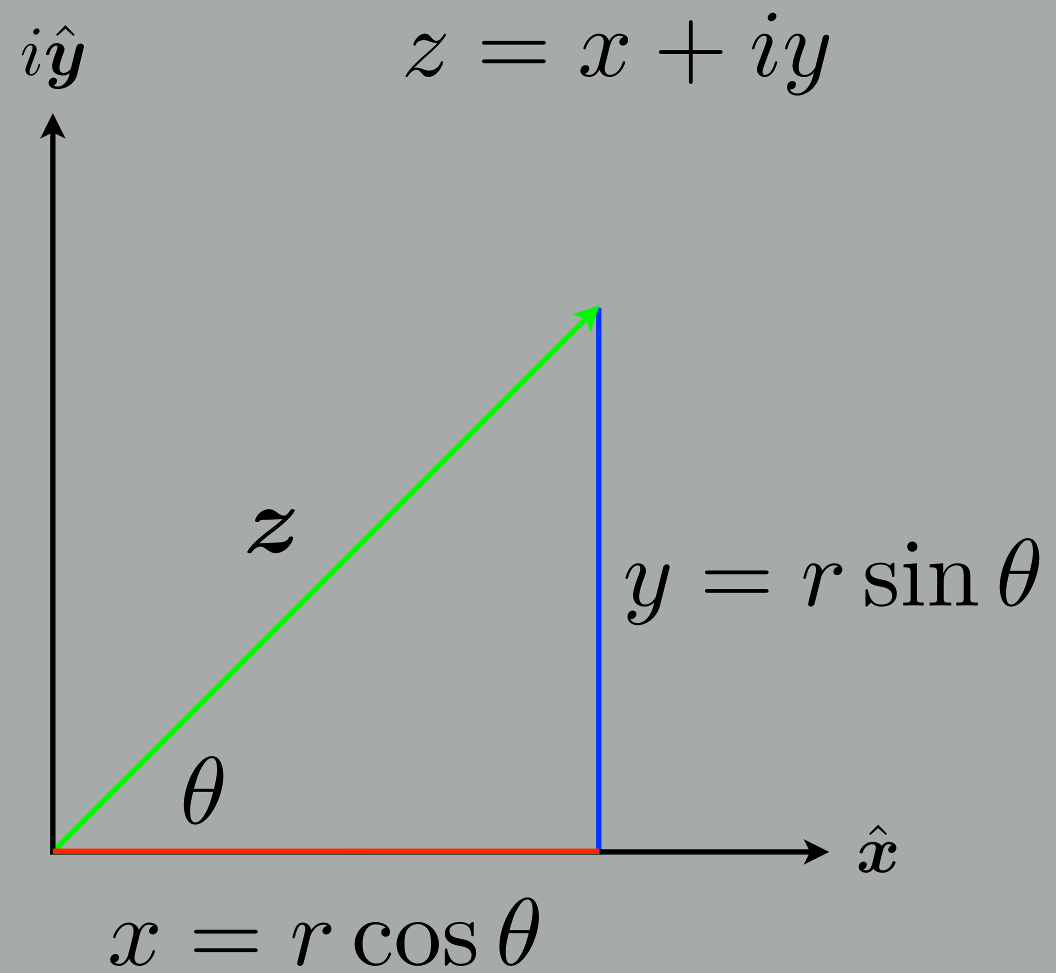
$$zz^* = (x+iy)(x-iy) = x^2+y^2$$

$$\|\boldsymbol{z}\| = \sqrt{\boldsymbol{z}^*\boldsymbol{z}}$$

$$\|\boldsymbol{z}\| = \sqrt{(re^{i\theta})(re^{-i\theta})} = r$$

The Complex Plane





$$|\boldsymbol{r}| = \sqrt{x^2 + y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = \sqrt{r^2}$$

$$|z| = \sqrt{(x+iy)(x-iy)} = \sqrt{x^2+y^2}$$

Magnitude and phase

Writing complex numbers $z = x + i\,y$ by using Euler's relation

$$z = re^{i\theta}$$

is very convenient since the magnitude is

$$|z| = \sqrt{zz^*} = \begin{cases} \sqrt{(z+iy)(z-iy)} = \sqrt{x^2+y^2} & \text{complex Cartesian} \\ \sqrt{re^{i\theta}re^{-i\theta}} = \sqrt{r^2} = r & \text{using Euler's relation} \end{cases}$$

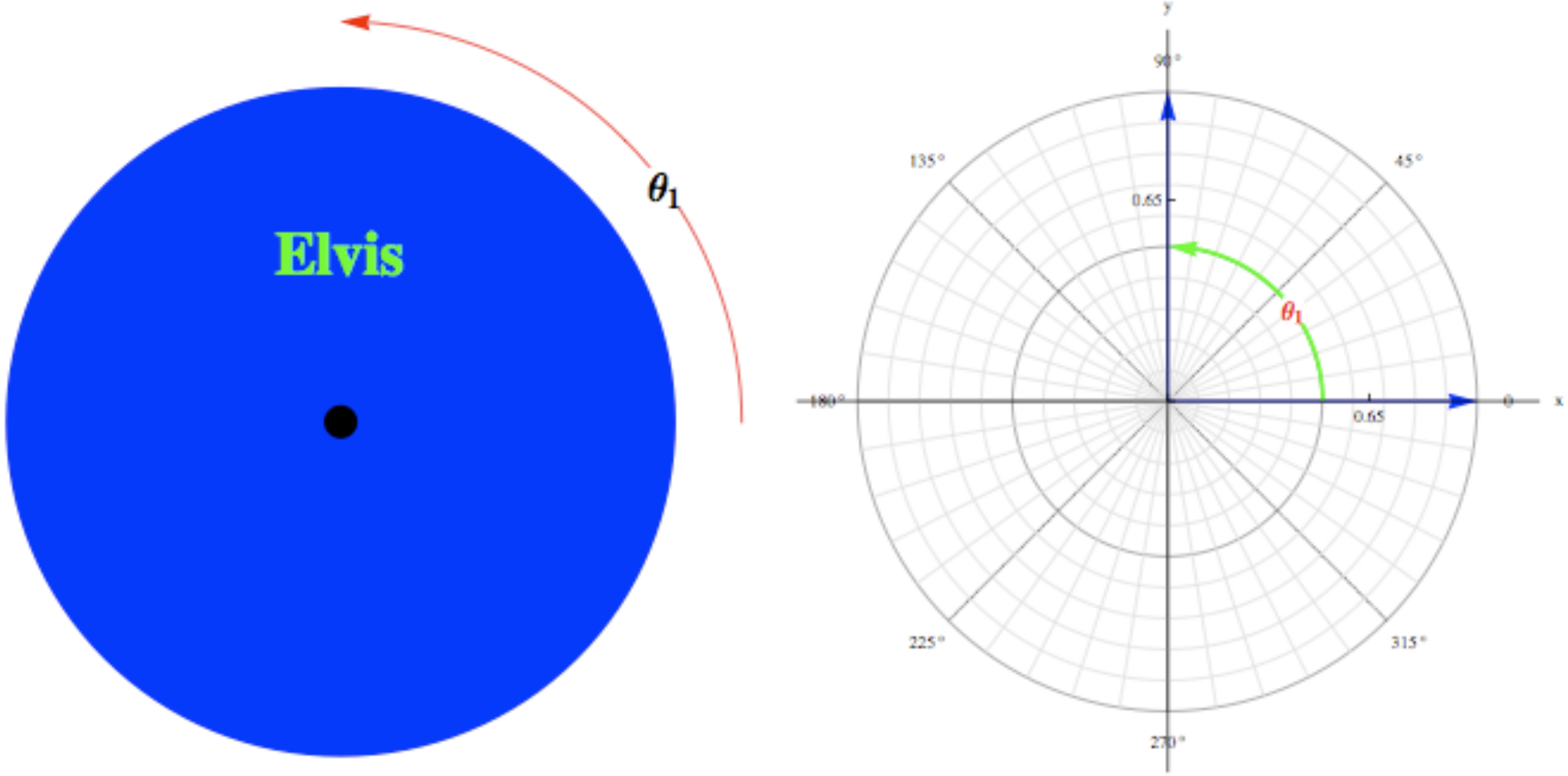
and the phase is

$$\angle|z| = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{complex Cartesian} \\ \theta = \arg z & \text{using Euler's relation} \end{cases}$$

symbol for angle

stands for "argument" and mean whatever multiplies i in exponent, which *is* the phase

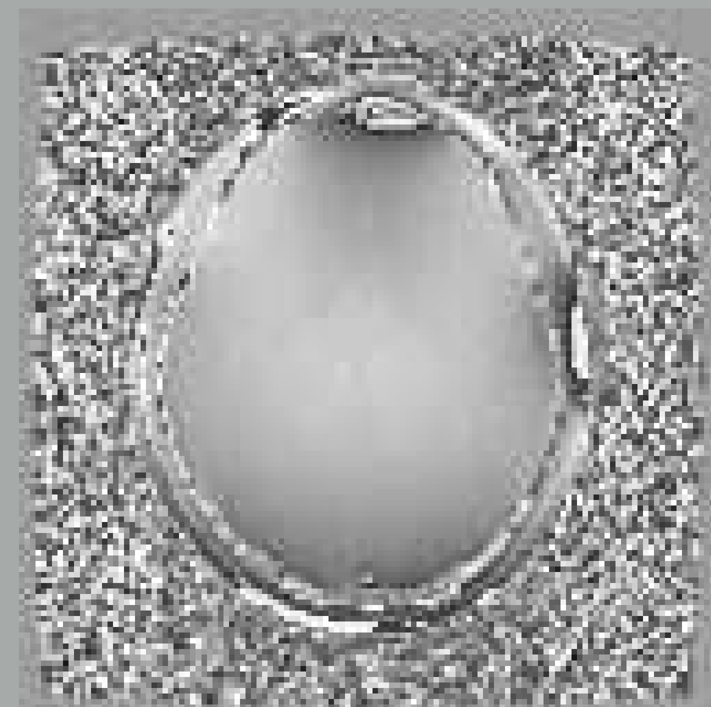
Phase angle



(a) The record spinning on the turntable. (b) The angle of rotation.

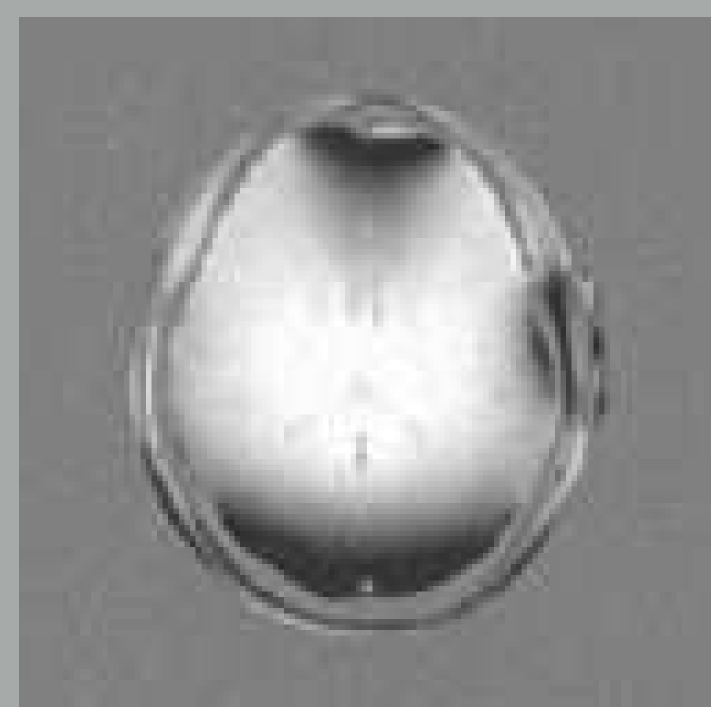
Example: Field Maps

magnitude



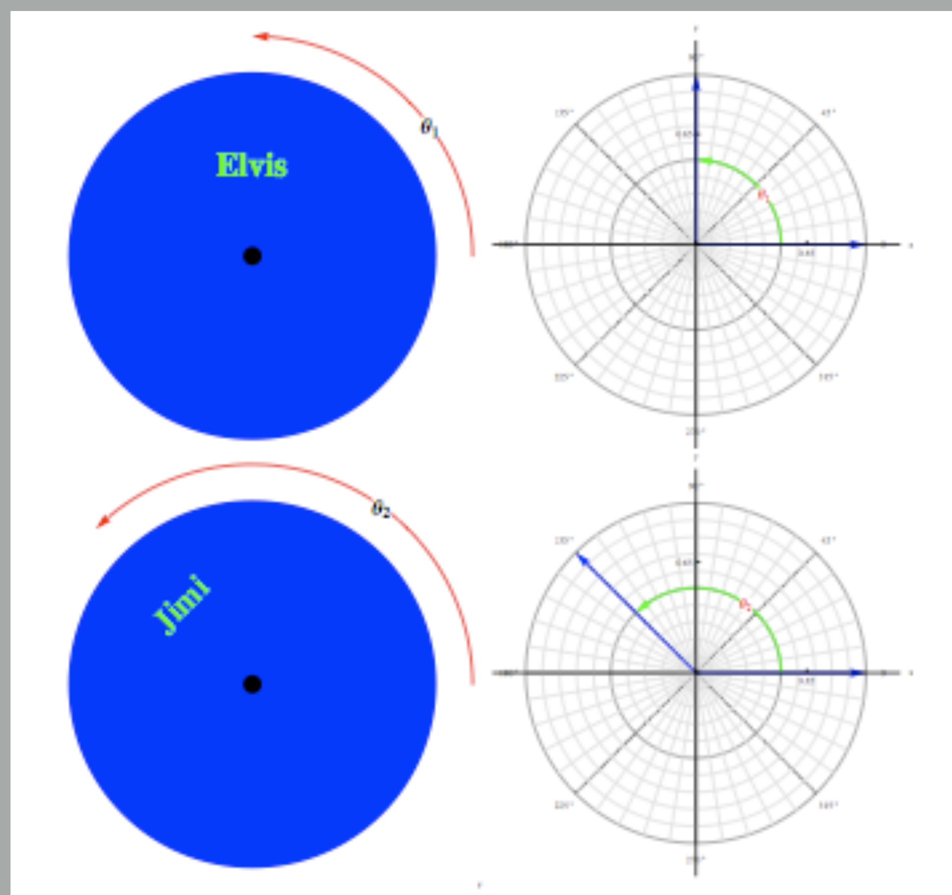
phase

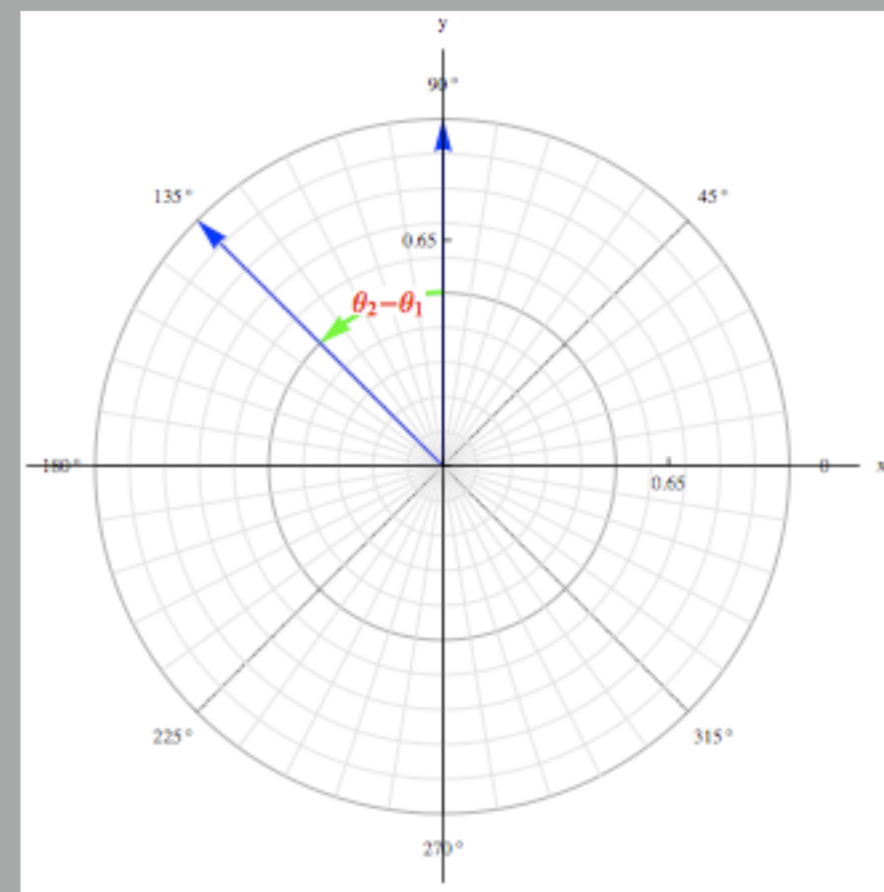
real



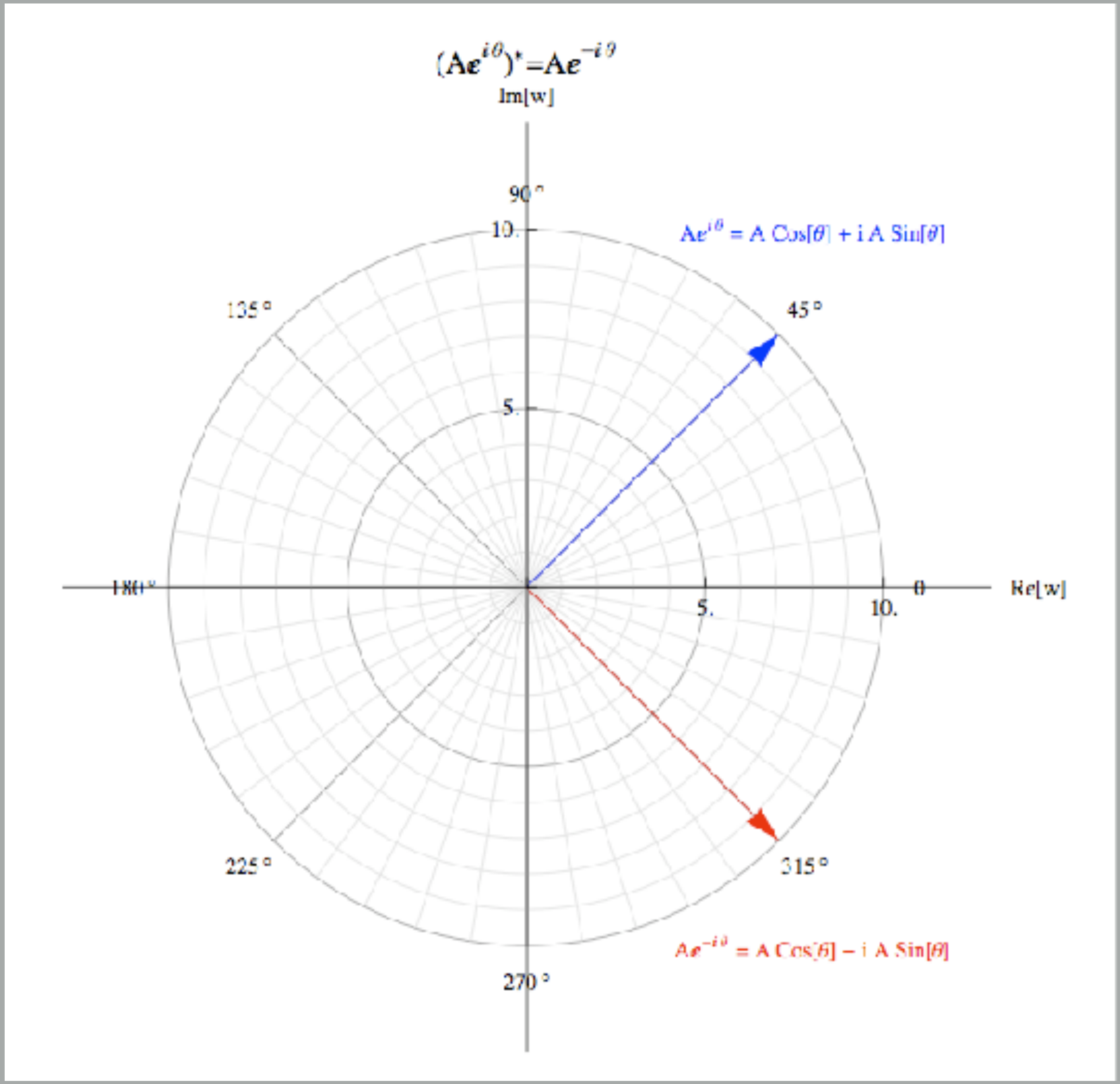
imaginary

Phase difference

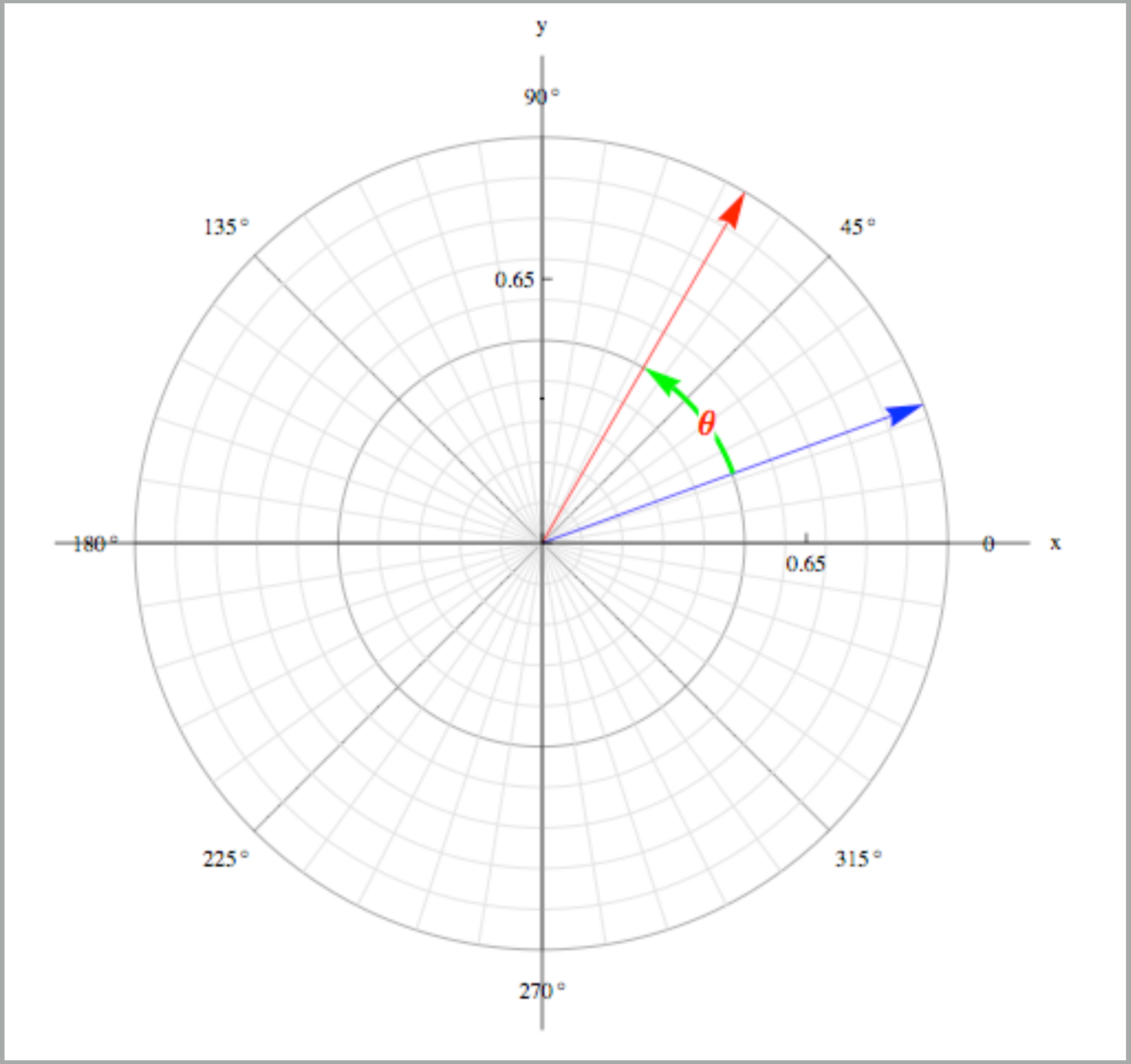




Complex Conjugation

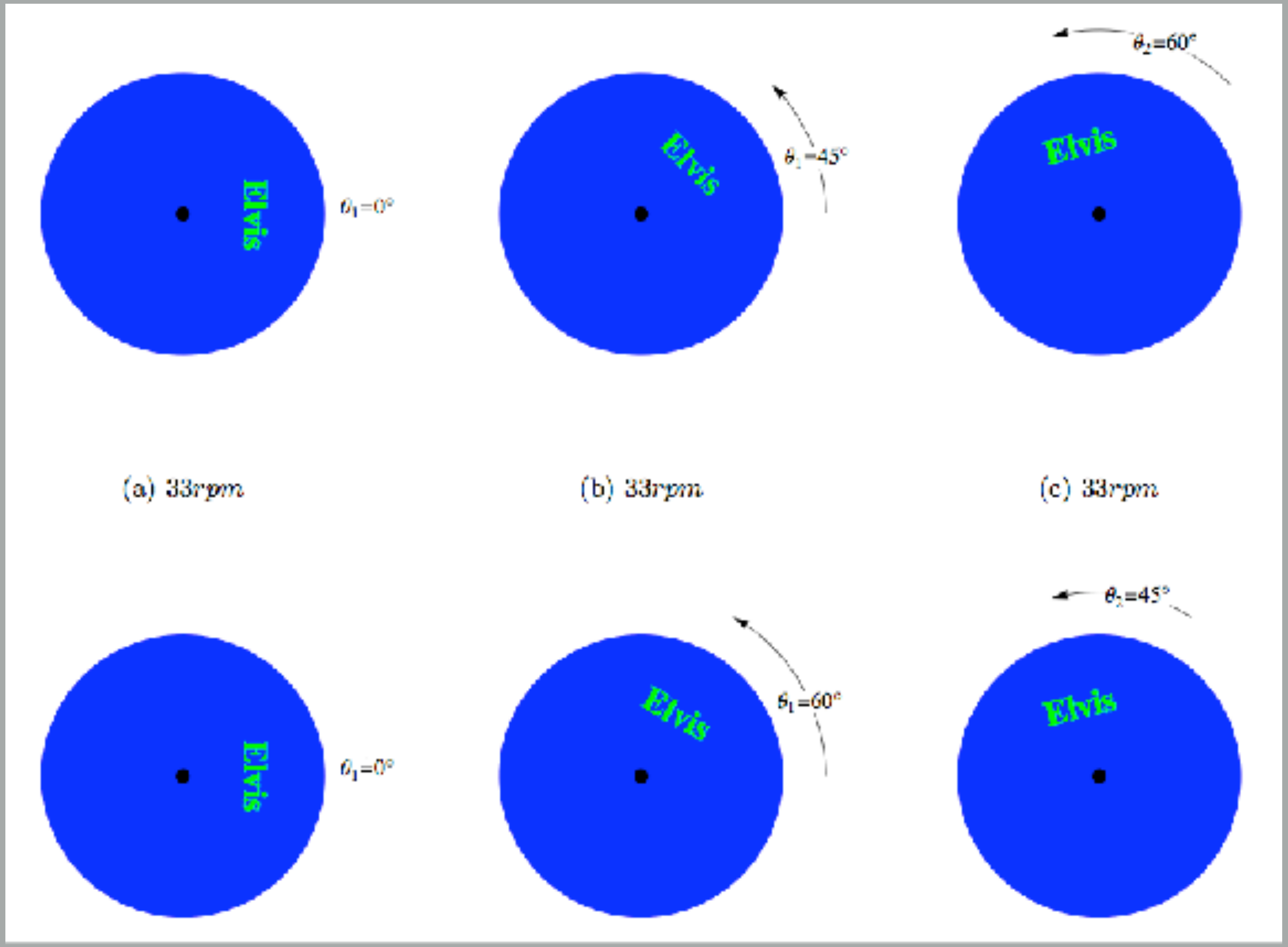


Rotations in 2D



$$\boldsymbol{u} = \boldsymbol{R}(\theta)\boldsymbol{v}$$

Order of Rotations in 2D



$$\boldsymbol{R}(\theta_1)\boldsymbol{R}(\theta_2) = \boldsymbol{R}(\theta_2)\boldsymbol{R}(\theta_1)$$

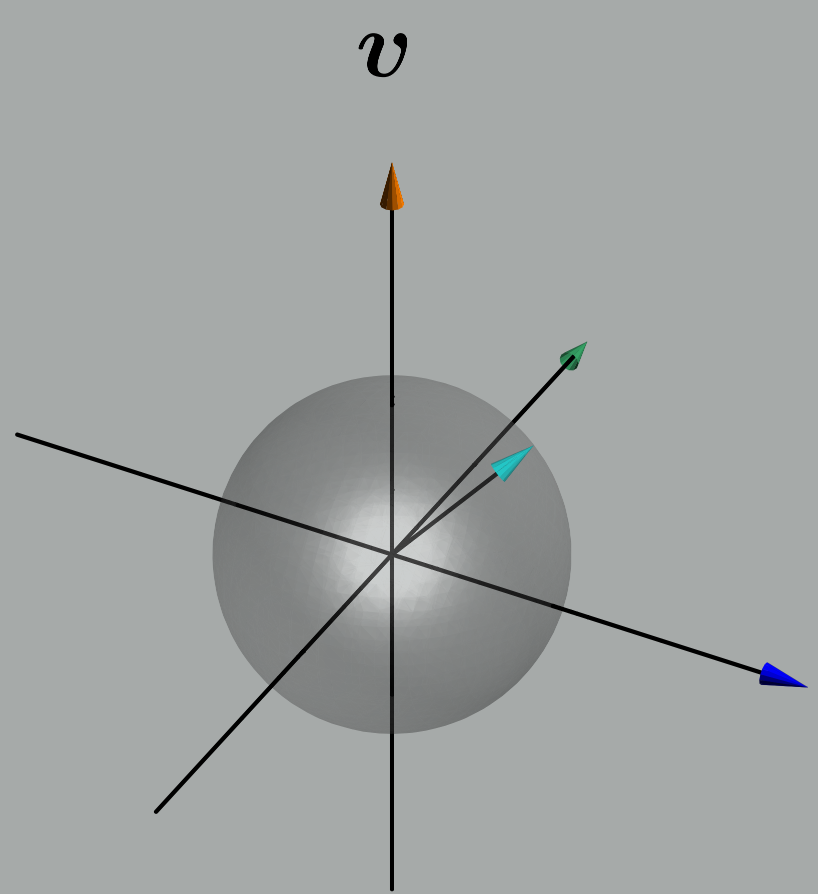
Order of Rotations in 2D

Order of rotations in 2D does not matter
i.e.,

$$\boxed{\text{rotations in 2D } \textit{commute}}$$

$$\boldsymbol{R}(\theta_1)\boldsymbol{R}(\theta_2) = \boldsymbol{R}(\theta_2)\boldsymbol{R}(\theta_1)$$

Vector rotations in 3D



Just as in 2D,

$$\boldsymbol{u} = \boldsymbol{R}\boldsymbol{v}$$

Where $\boldsymbol{R}$ is now a 3 x 3 rotation matrix

Rotation Matrices in 3D

$$\boldsymbol{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

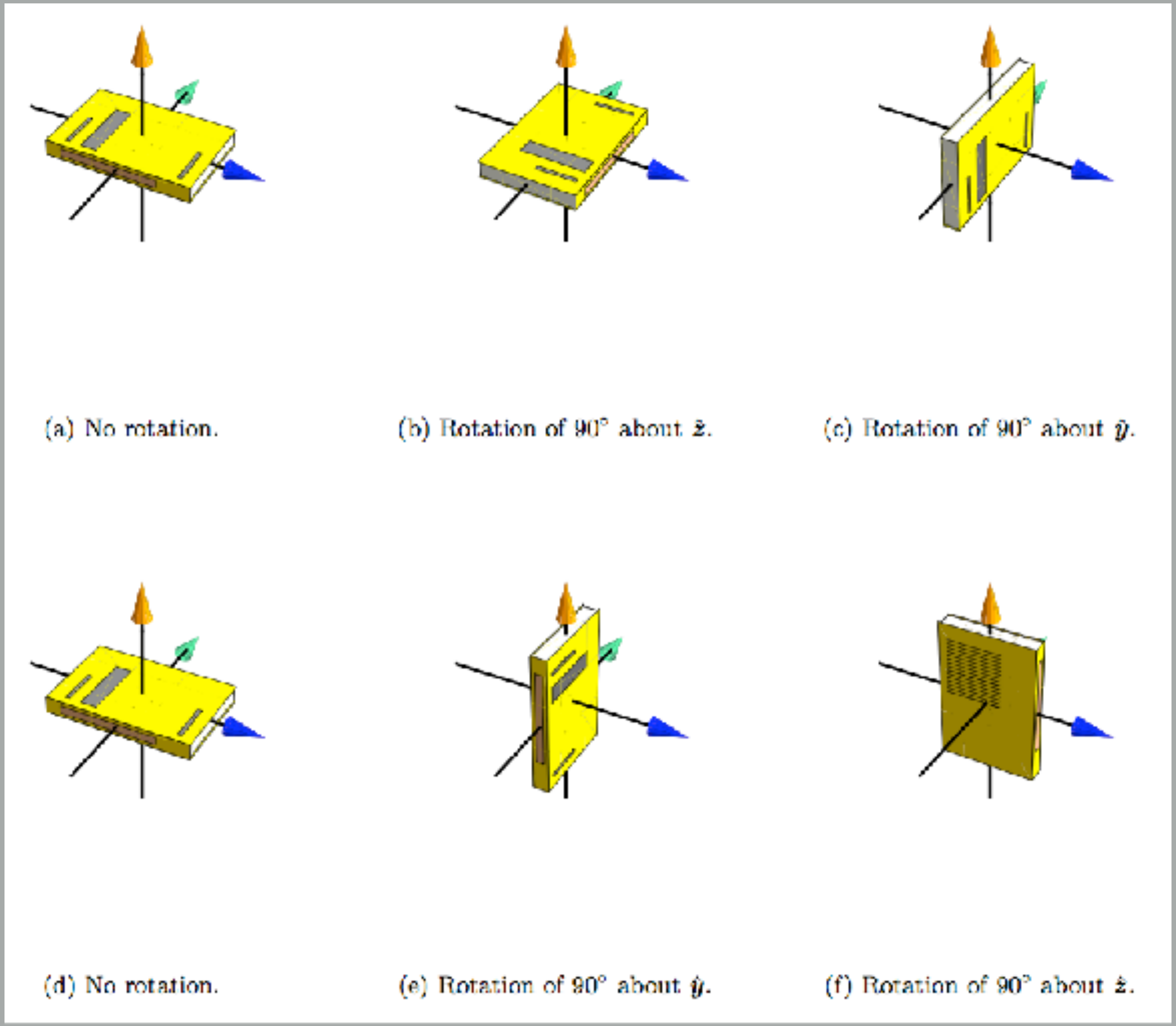
$$\boldsymbol{R}_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$\boldsymbol{R}_z(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Object Rotations in 3D

But *object* rotations are another story...

Order of Rotations in 3D



$$\boldsymbol{R}(\theta_1)\boldsymbol{R}(\theta_2) \neq \boldsymbol{R}(\theta_2)\boldsymbol{R}(\theta_1)$$

Order of Rotations in 3D

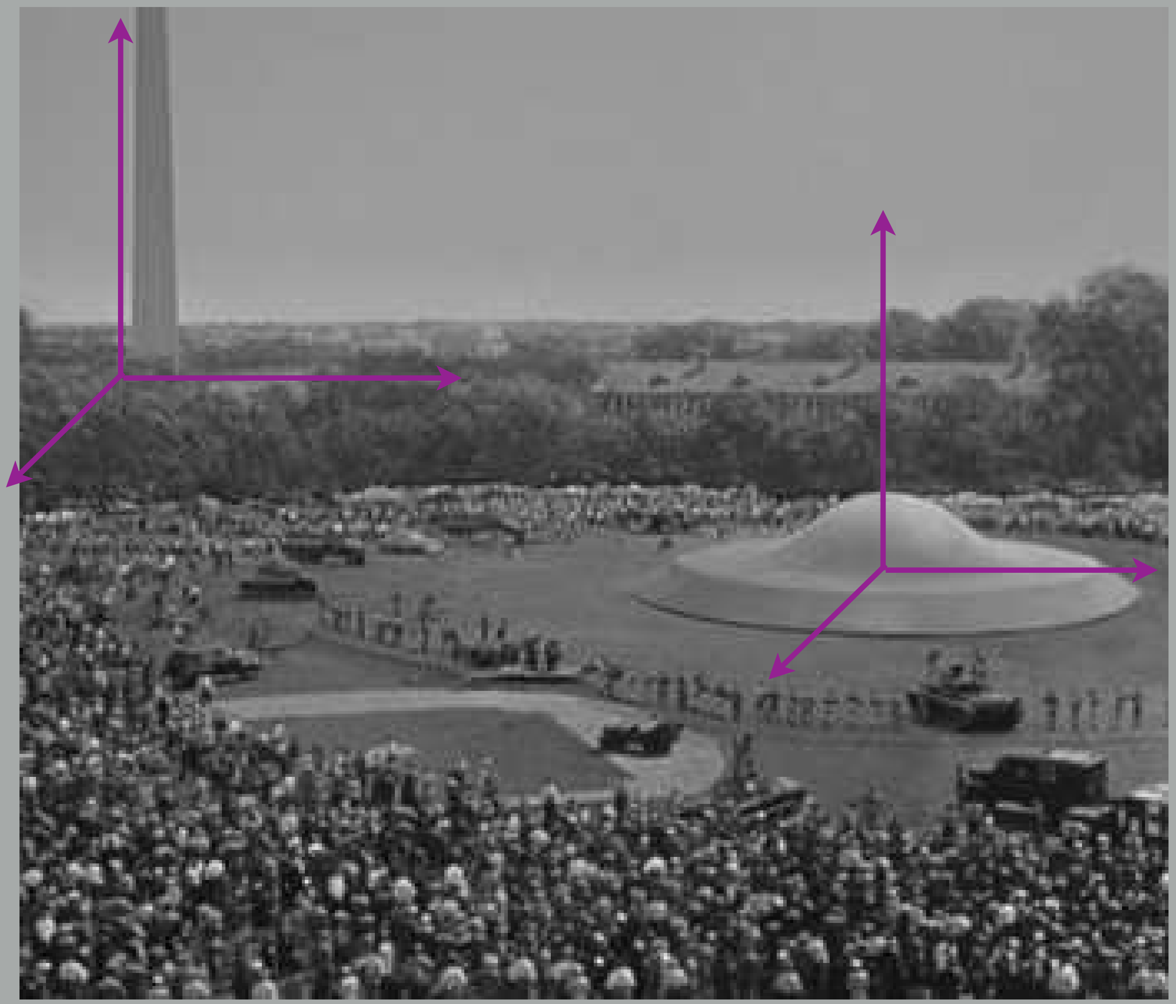
Order of rotations in 3D *does* matter

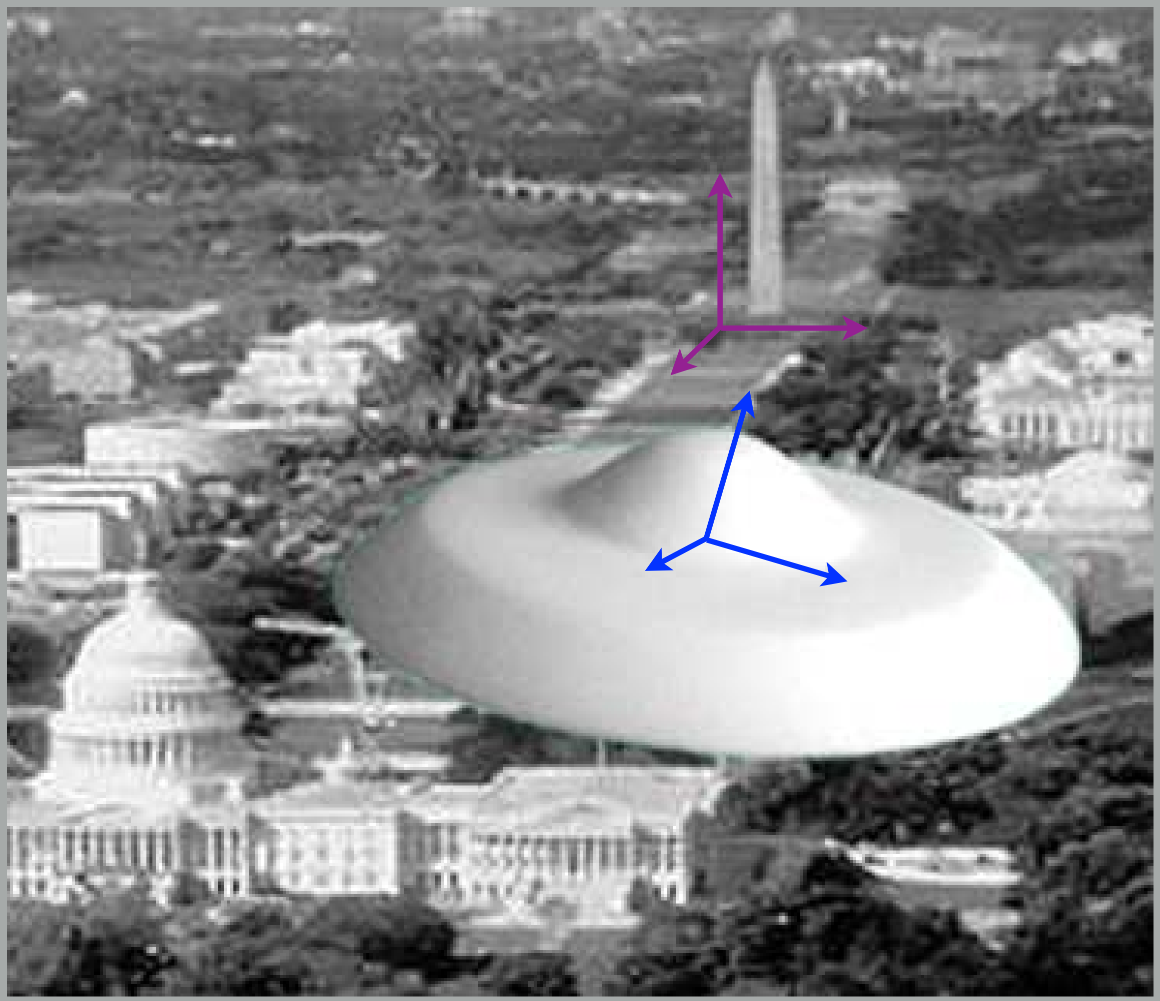
i.e.,

rotations in 3D do *not* necessarily commute

$$\boldsymbol{R}(\theta_1)\boldsymbol{R}(\theta_2) \neq \boldsymbol{R}(\theta_2)\boldsymbol{R}(\theta_1)$$

Describing Rotations in 3D



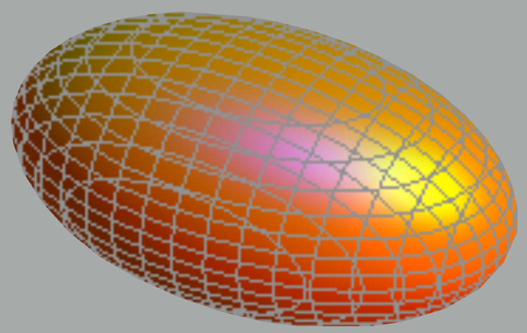


What's different about 3D?

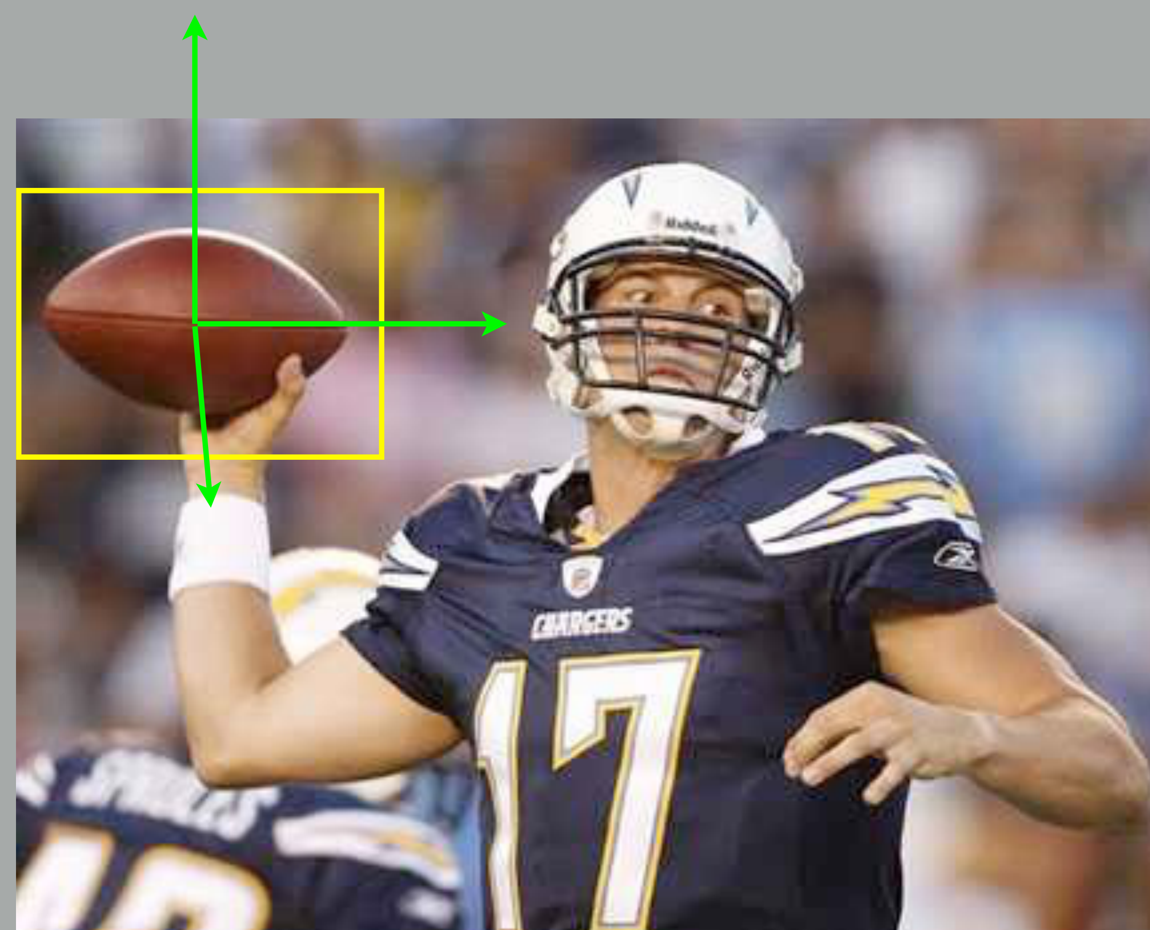
Lab (field) coordinates

What's different about 3D?

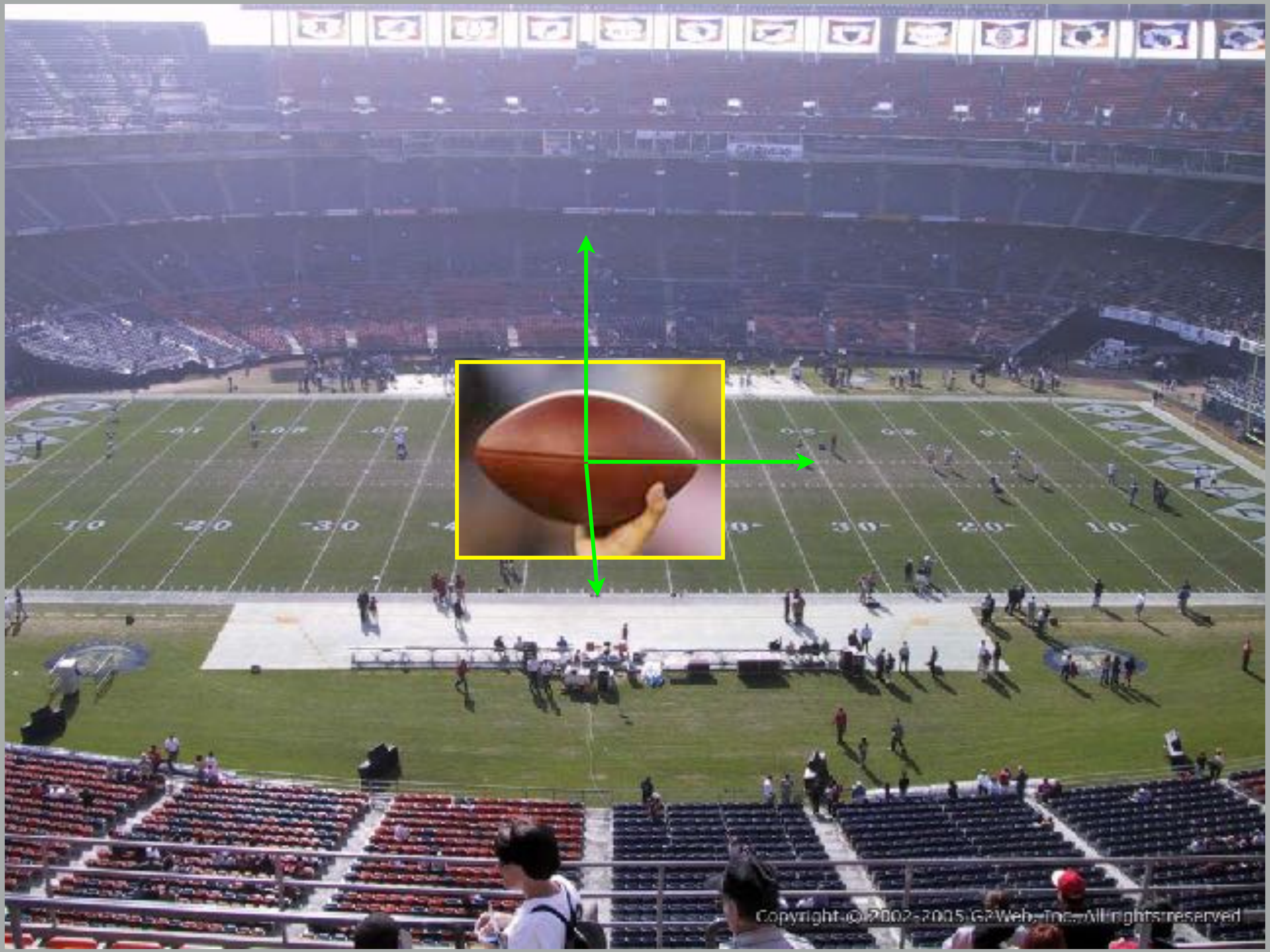
$\approx$



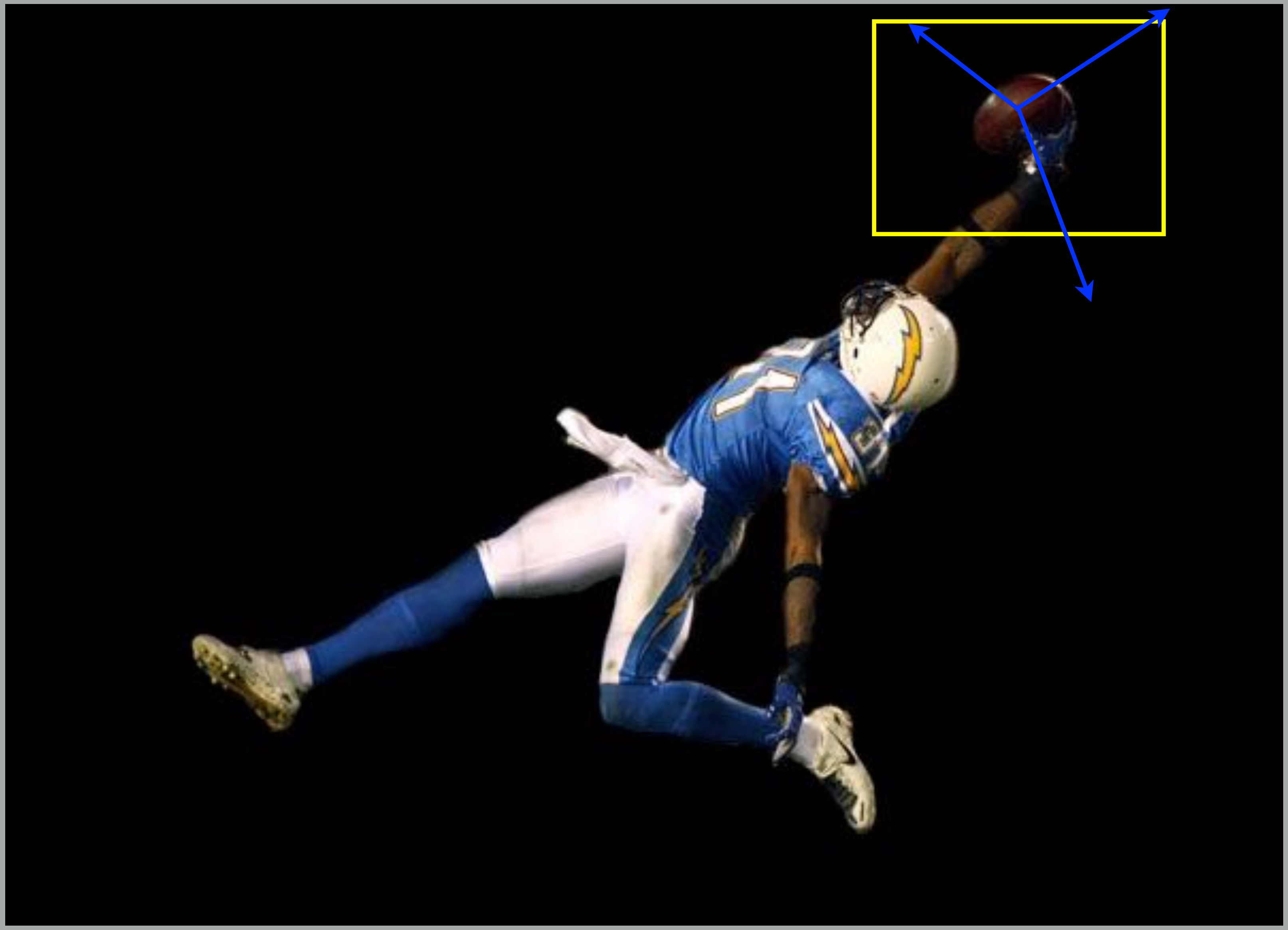
What's different about 3D?



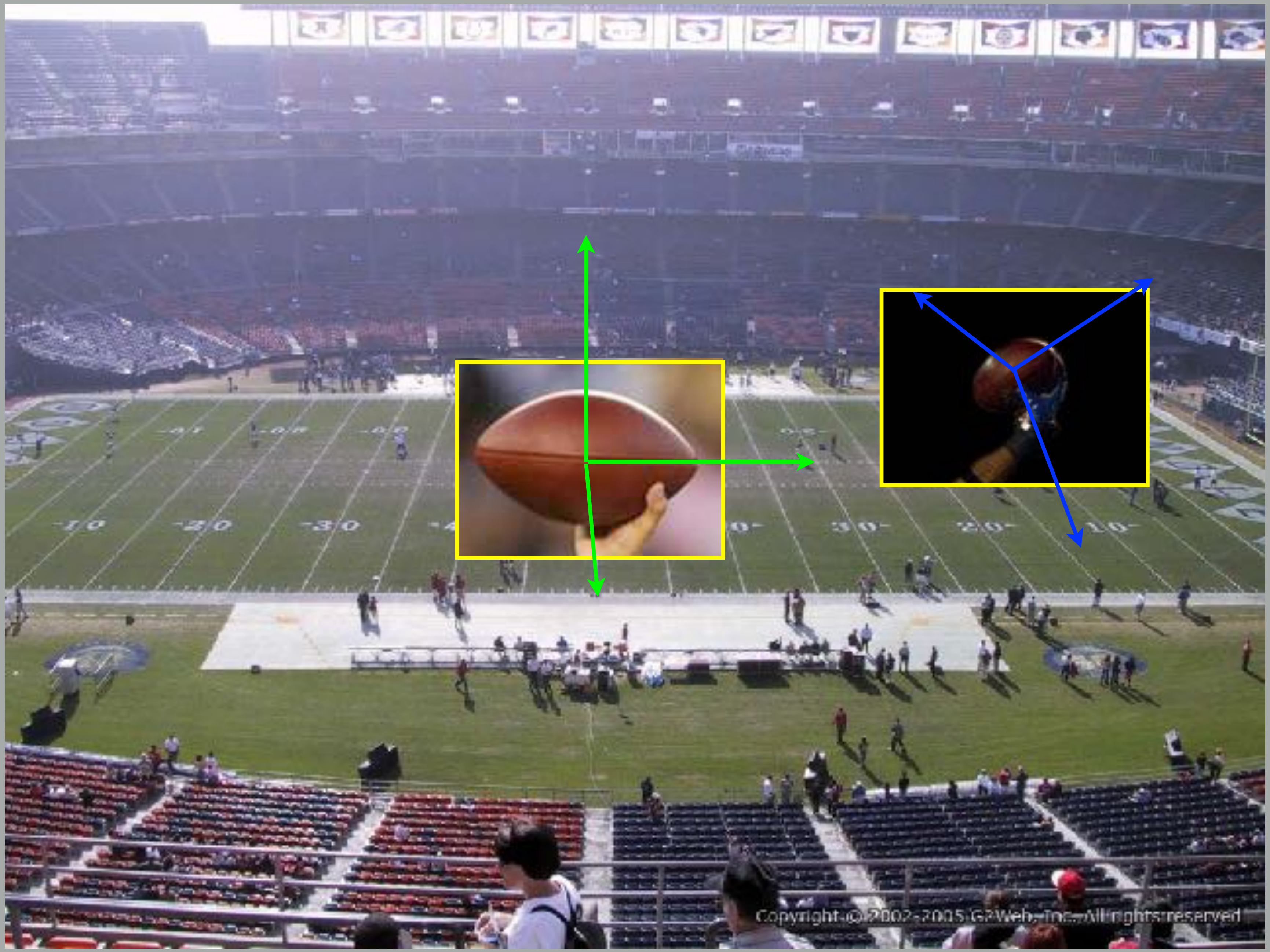
Lab (field) coordinates



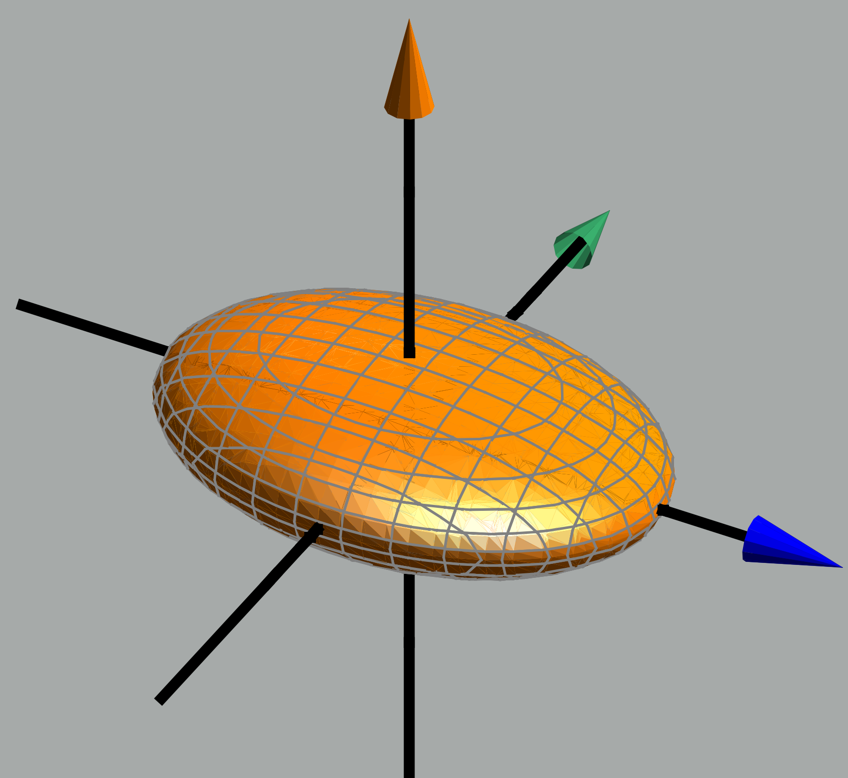
What's different about 3D?



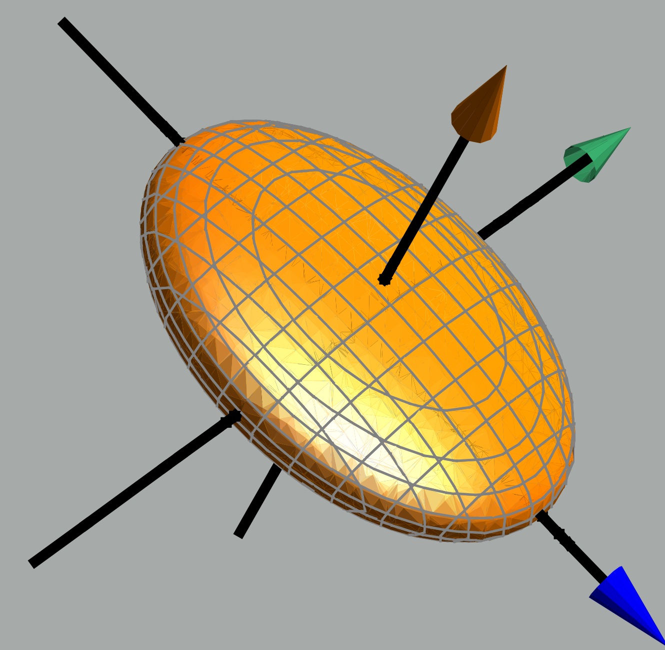
Player coordinates



Describing Rotations in 3D

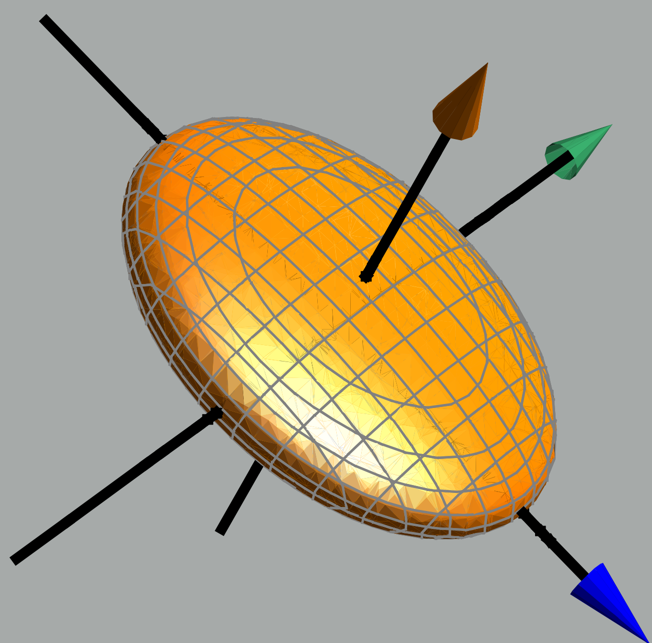


S

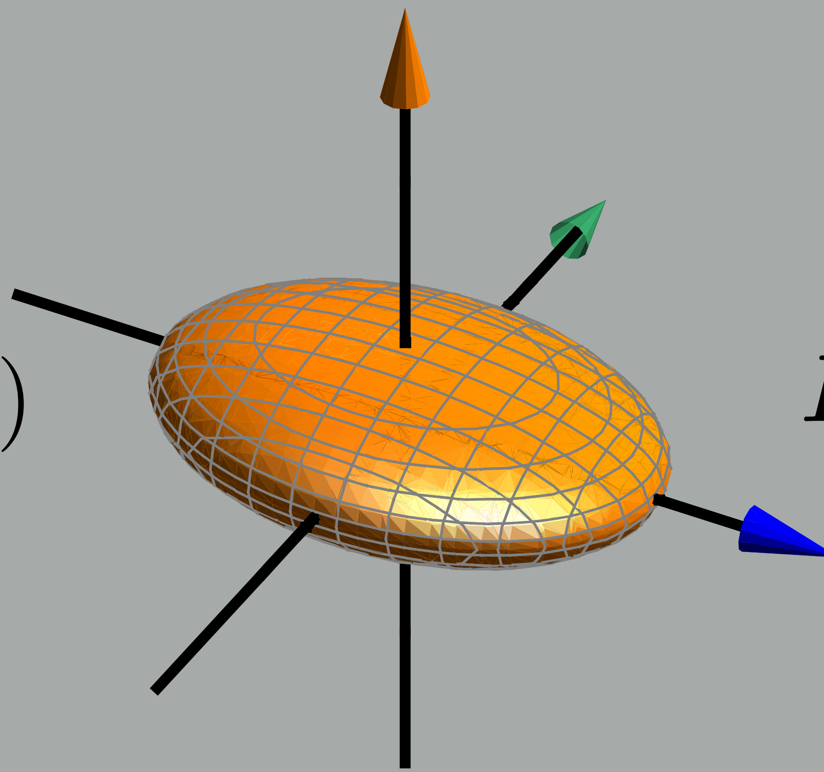


S'

Similarity Transform



$= \boldsymbol{R}^{-1}(\vartheta, \varphi)$



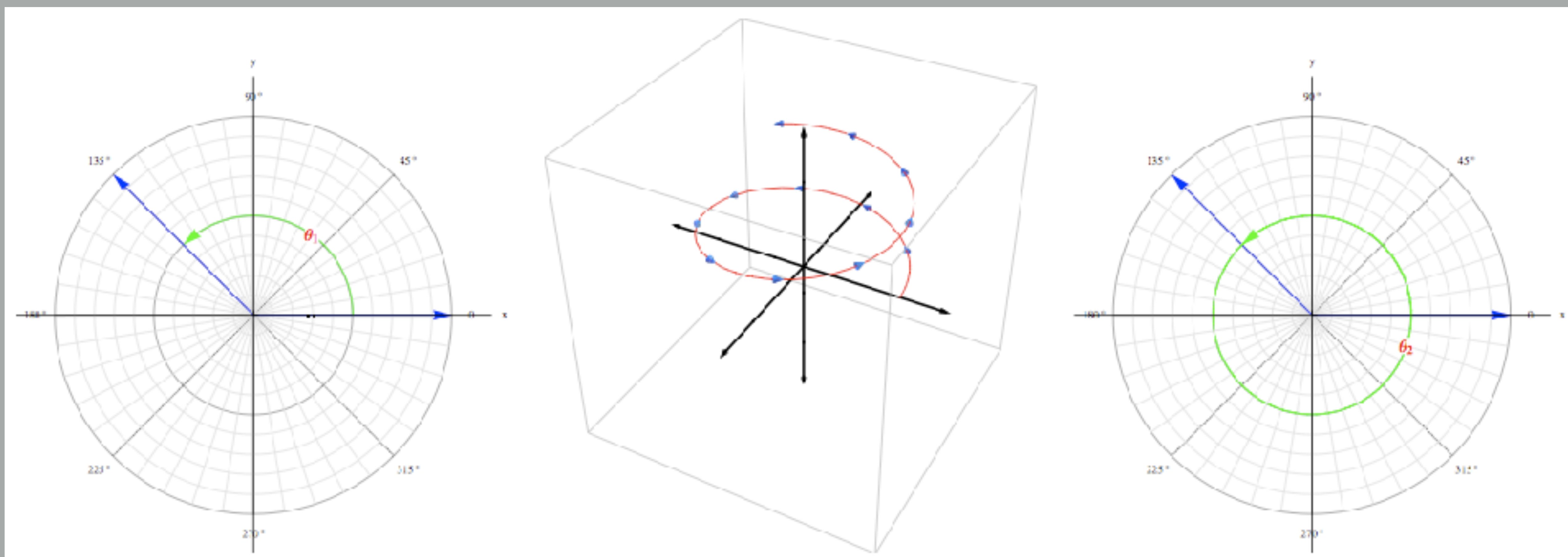
$\boldsymbol{R}(\vartheta, \varphi)$

$$\boxed{\boldsymbol{S}' = \boldsymbol{R}^{-1}\boldsymbol{S}\boldsymbol{R}}$$

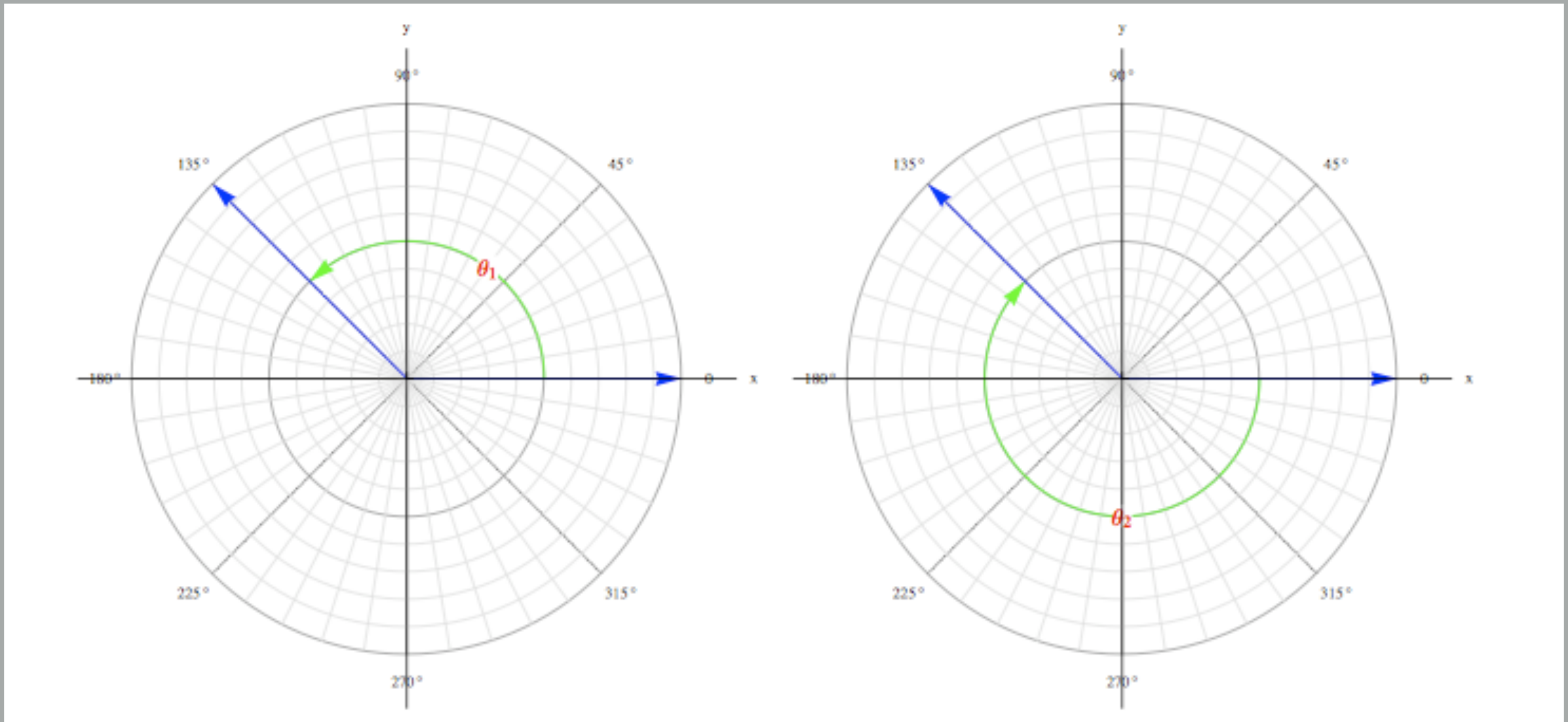
Looks just the same as in 2D!

Extra slides

Phase Wrapping

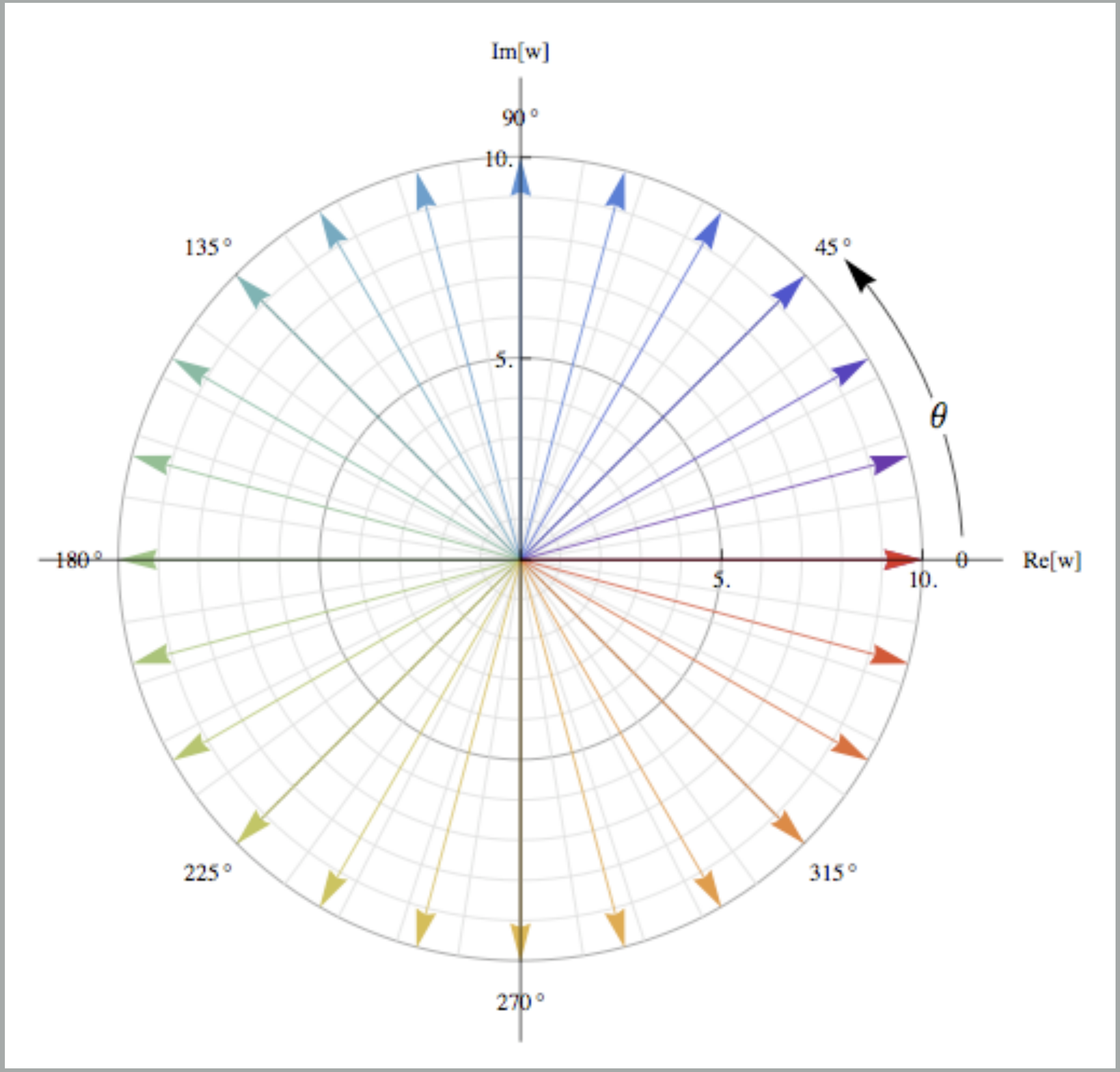


Aliasing

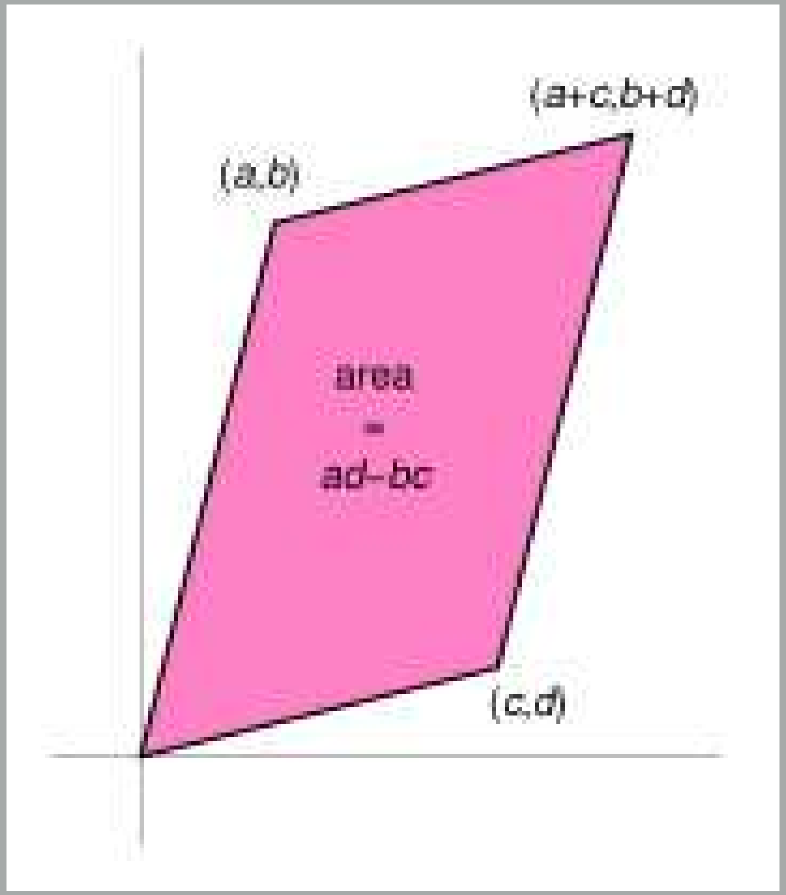


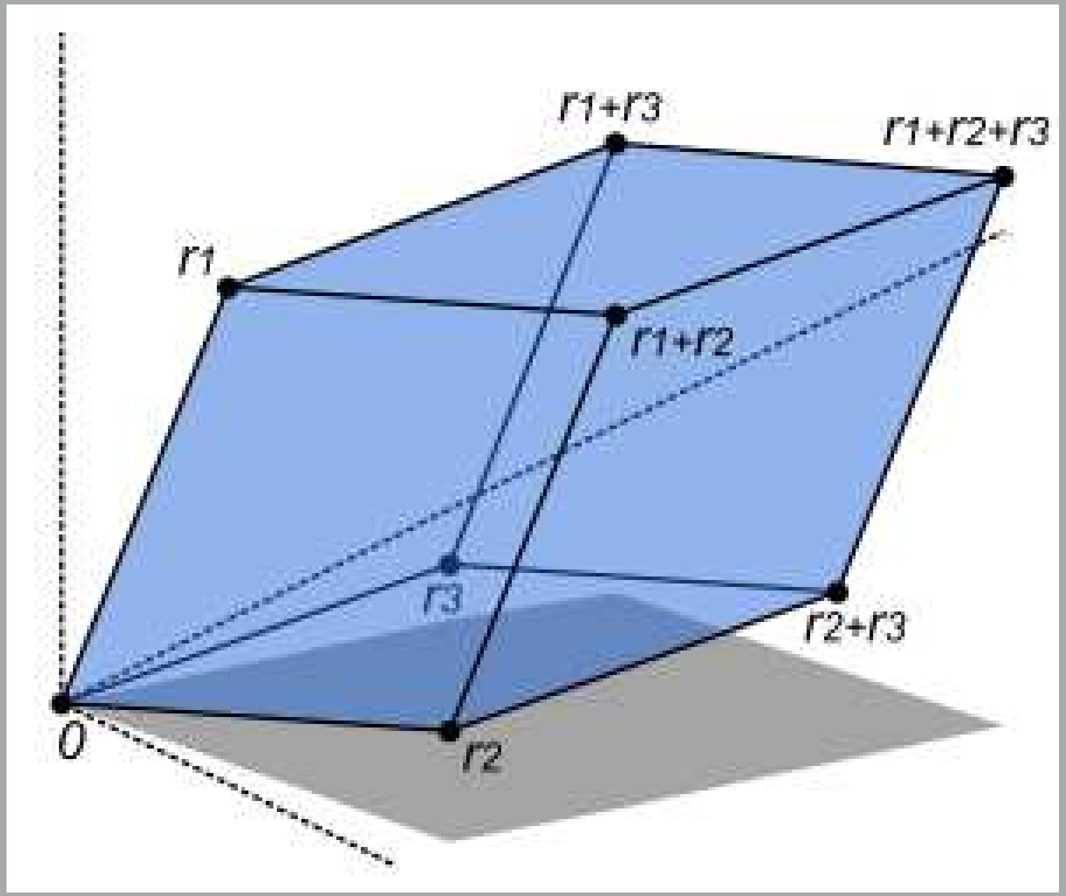
y
90°
135°
45°
θ_1
180°
0
x
225°
315°
270°
y
90°
135°
45°
180°
0
x
θ_2
225°
315°
270°

The Complex Plane



Matrix Determinant





$$|\boldsymbol{A}| = area$$

$$\boldsymbol{A} = 2 \times 2 \text{ real matrix}$$

$$|\boldsymbol{A}| = volume$$

$$\boldsymbol{A} = 3 \times 3 \text{ real matrix}$$

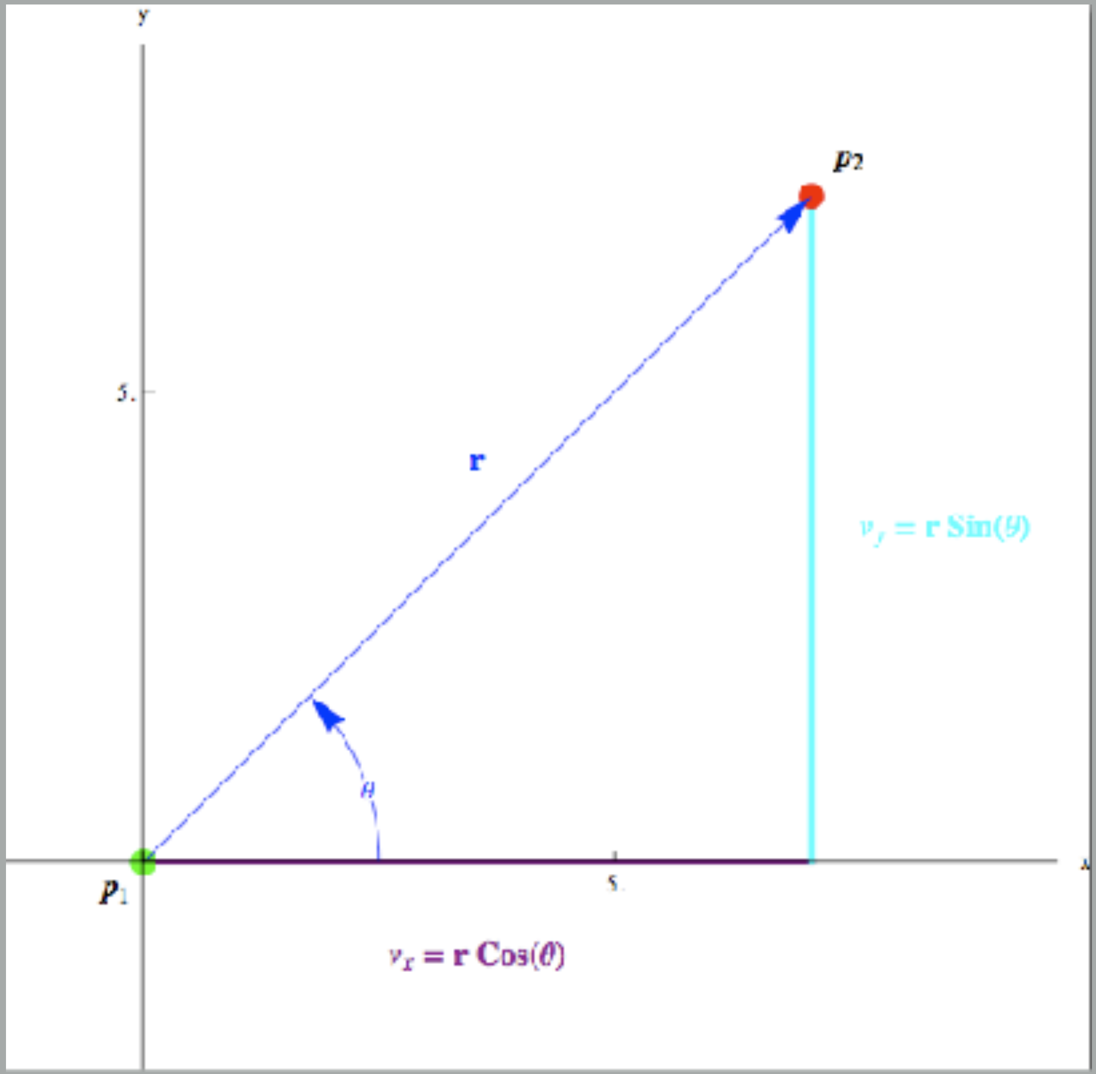
Matrix Multiplication

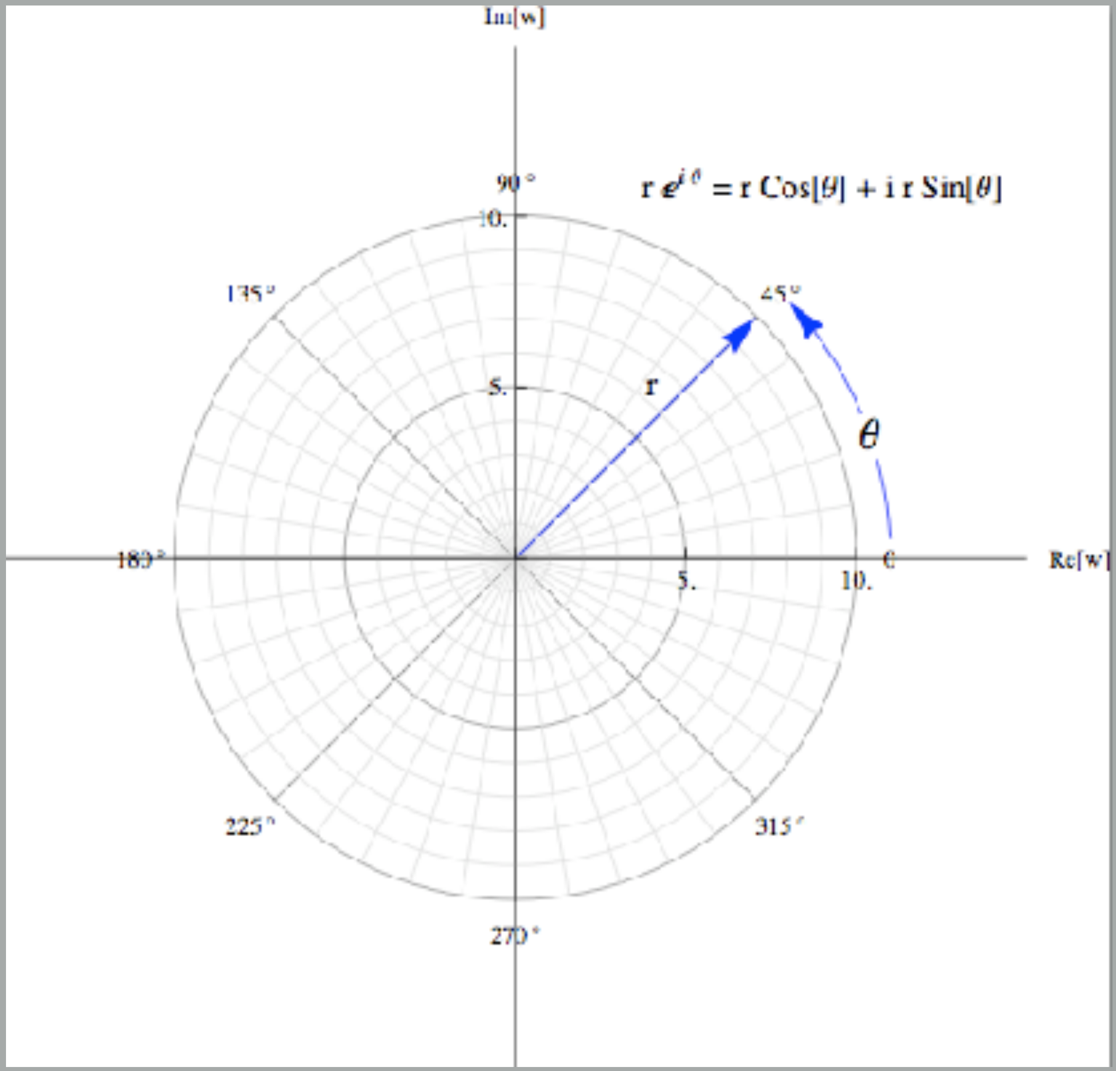
Dot Product

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21} \quad a_{22} \quad a_{23}} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & \boxed{c_{12}} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

$$c_{12} = \boxed{\begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix}} \boxed{\begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix}} = \boxed{a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}}$$

Complex Numbers





Vector Multiplication

$$\boldsymbol{q} = (a\ ,\ b)$$ *columns* here ....

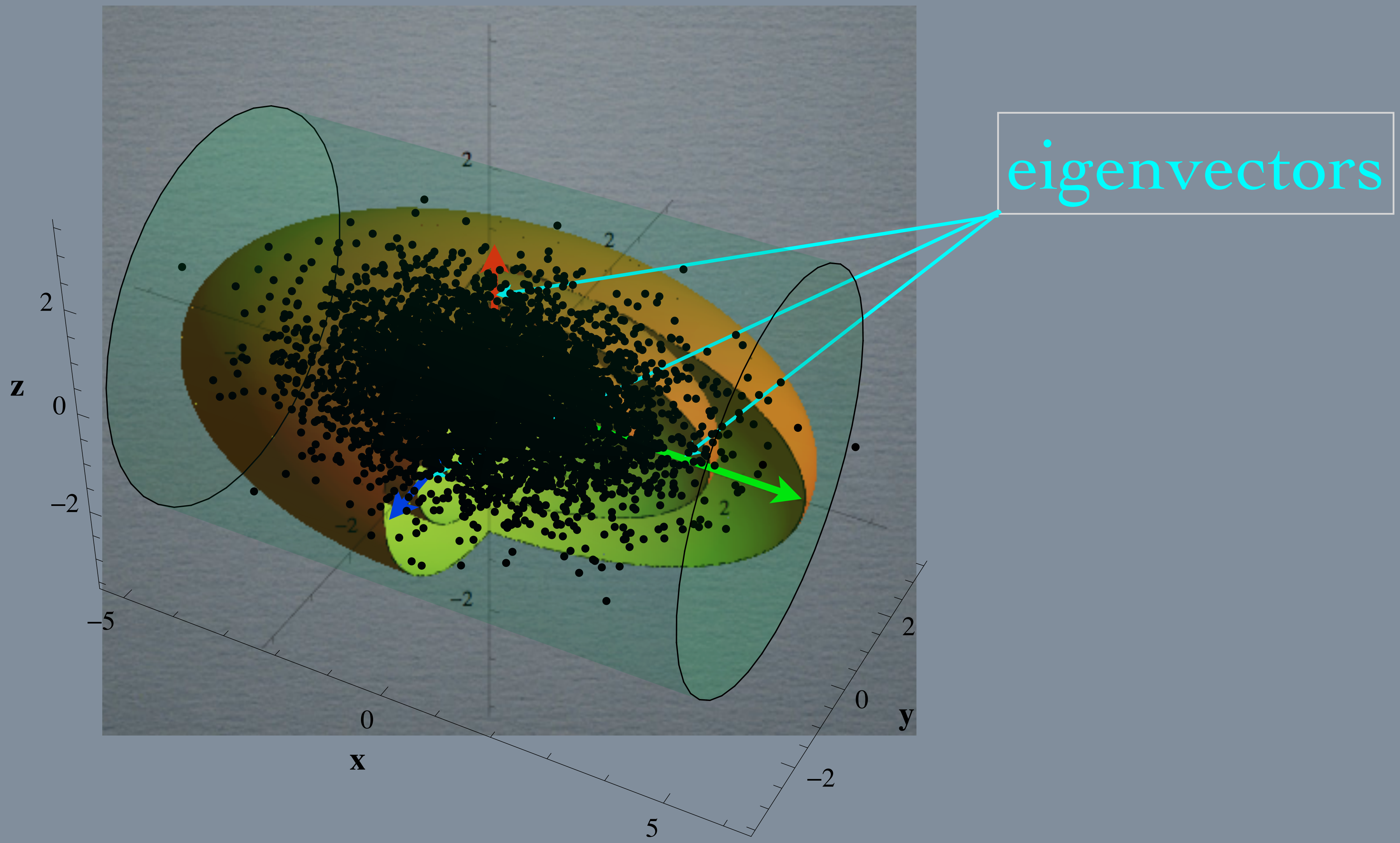
$$\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$ must equal *rows* here

$$\boldsymbol{qr} = (a, b)\begin{pmatrix} x \\ y \end{pmatrix} = ax + by$$

Slides from previous lecture to add here

START

Diffusion Anisotropy in 3D

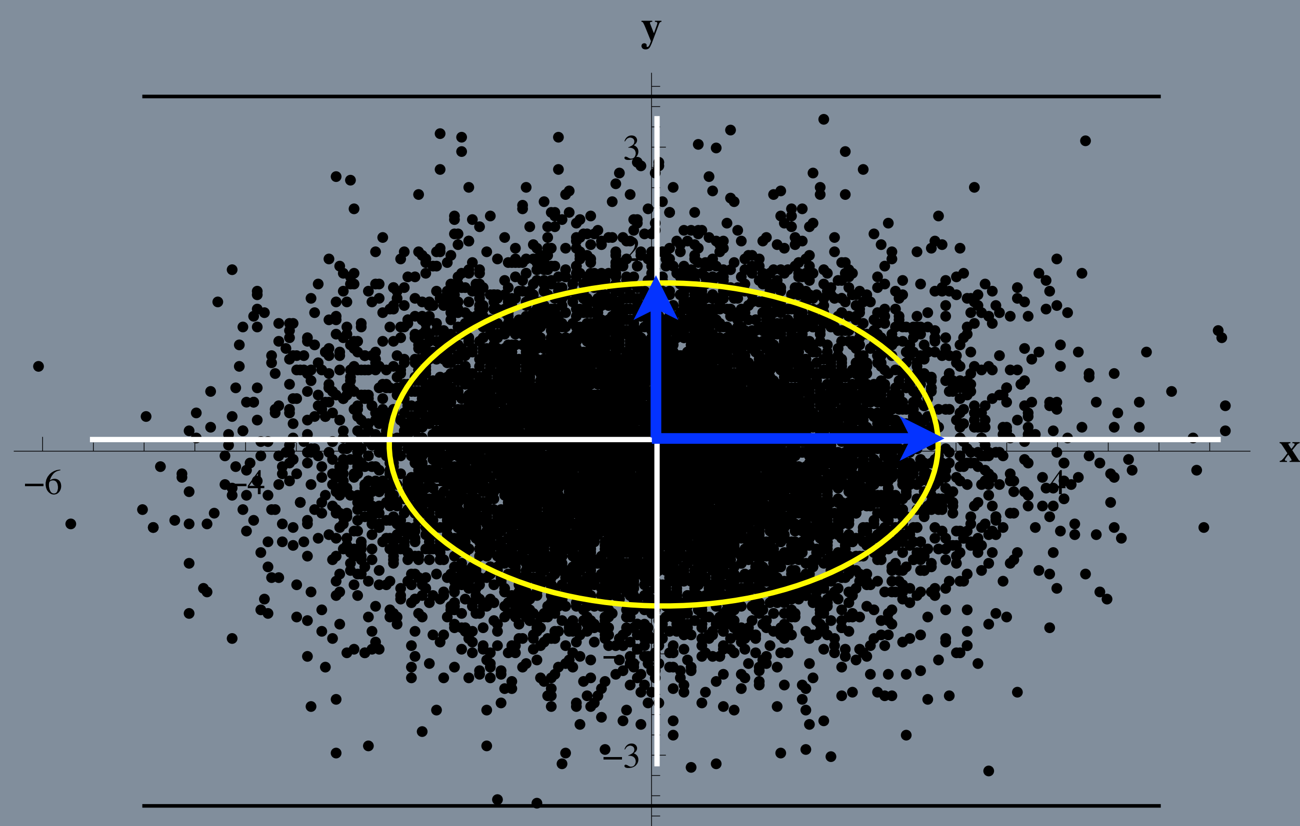


probability contours in 3D

The 2D Gaussian Distribution

$$P(\boldsymbol{r}|\boldsymbol{r}_0, \tau) \sim N(\boldsymbol{r}_0, \boldsymbol{\Sigma})$$

$$\boldsymbol{r} = \{x, y\}$$



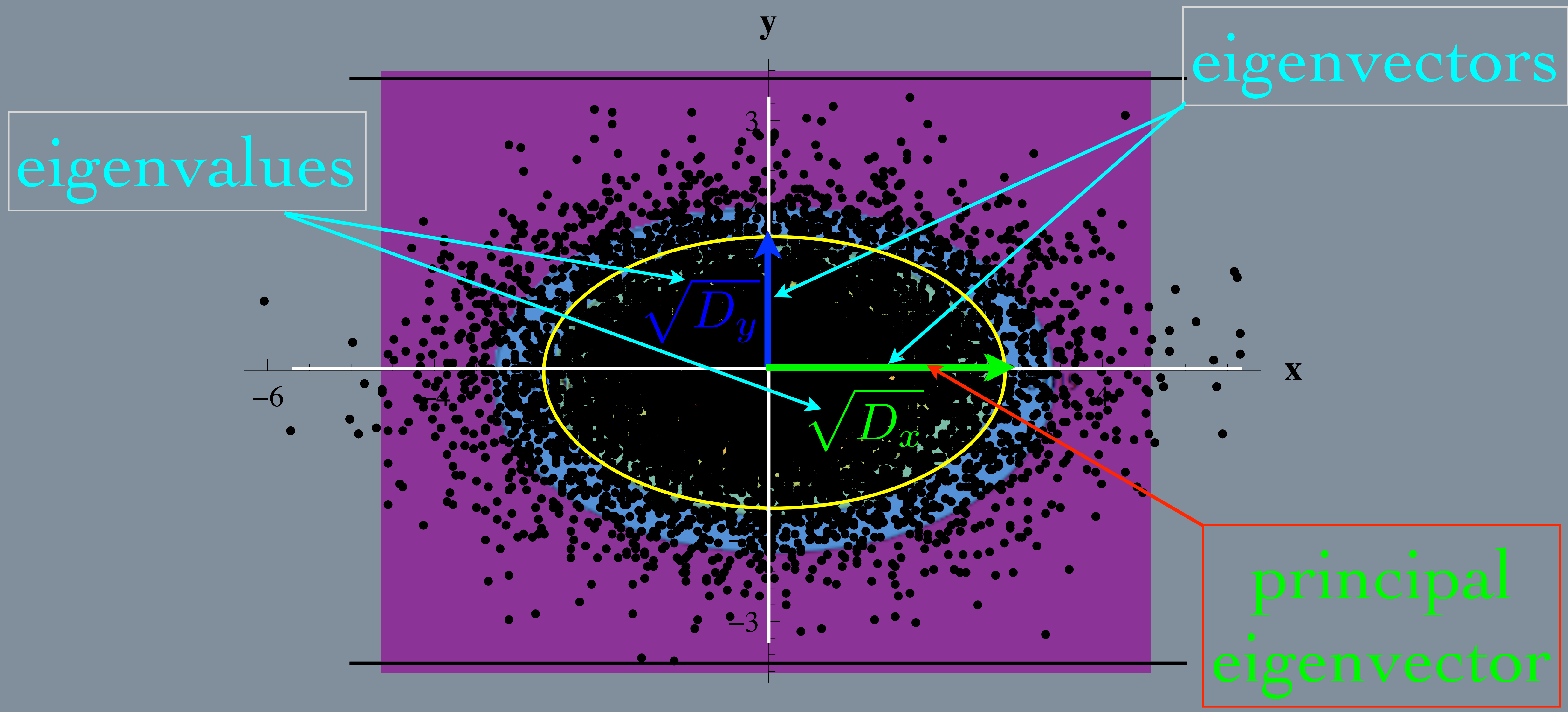
Covariance matrix

Diffusion Tensor

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} = 4\tau \begin{pmatrix} D_x & 0 \\ 0 & D_y \end{pmatrix}$$

$\{D_x, D_y\}$ are the *principal diffusivities*

Anisotropic Gaussian Diffusion



1. The relative dimensions of the contours tells us about local structure

2. The orientation of the largest dimension is related to the orientation of the structure

Covariance Matrix

Motivation for next lecture

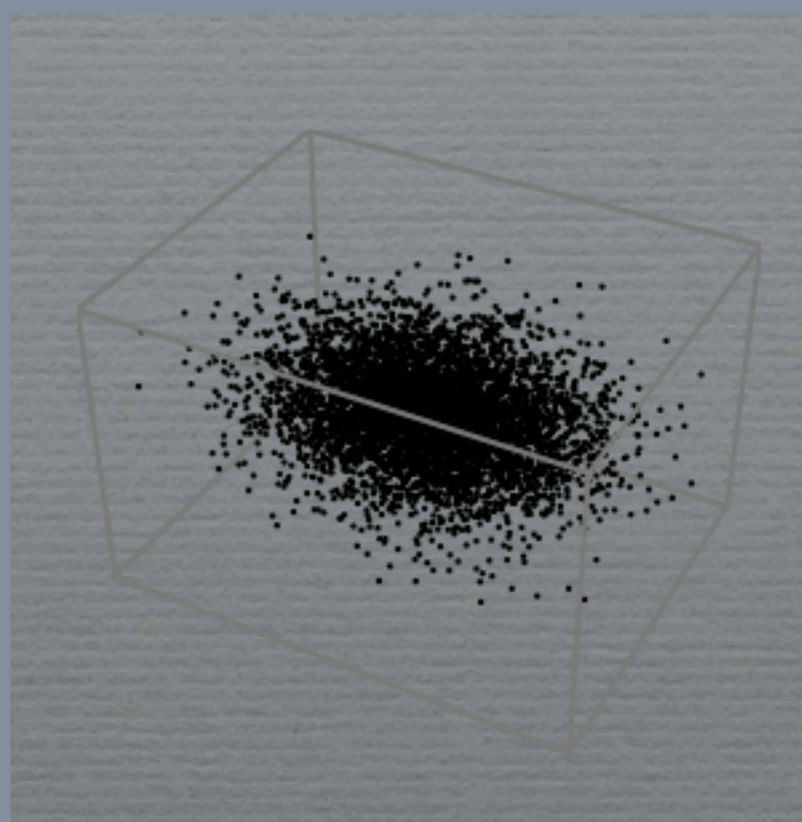
$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{xx}^2 & \sigma_{xy}^2 \\ \sigma_{yx}^2 & \sigma_{yy}^2 \end{pmatrix} = 4\tau \underbrace{\begin{pmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{pmatrix}}_{\boldsymbol{D}}$$

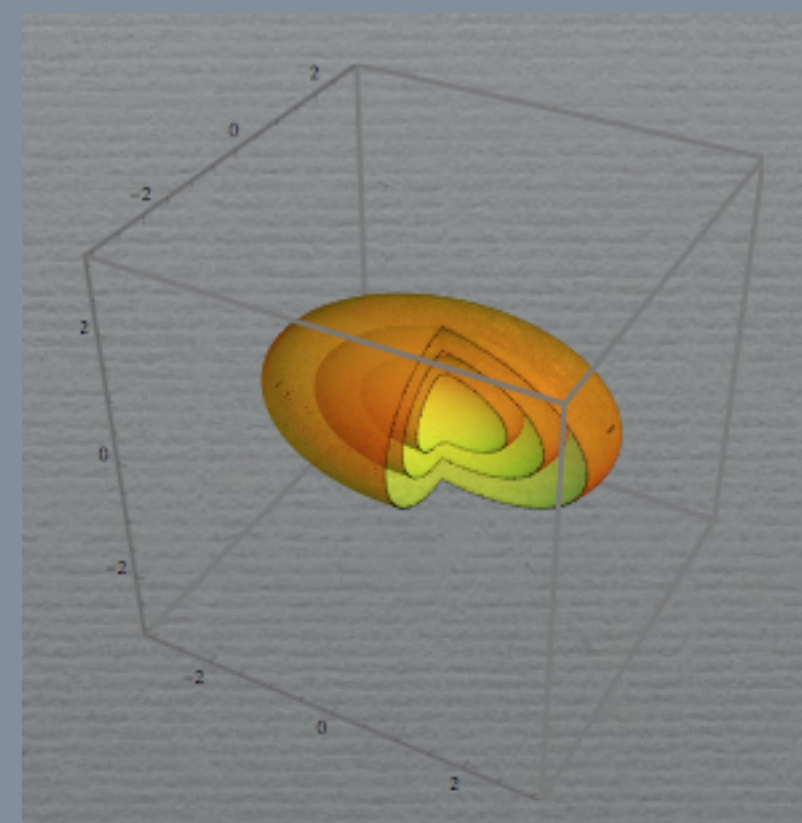
diffusion tensor

$$\boldsymbol{D} = \begin{pmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{pmatrix}$$

3D Gaussian Distribution

$$P(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mu)^t\Sigma^{-1}(\mathbf{x}-\mu)\right]$$





Covariance Matrix

Motivation for next lecture

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{xx}^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\ \sigma_{yx}^2 & \sigma_{yy}^2 & \sigma_{yz}^2 \\ \sigma_{zx}^2 & \sigma_{zy}^2 & \sigma_{zz}^2 \end{pmatrix} = 6\tau \underbrace{\begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix}}_{\boldsymbol{D}}$$

diffusion tensor

symmetric and real

$$\boldsymbol{D} = \begin{pmatrix} D_{xx} & \boxed{D_{xy}} & \boxed{D_{xz}} \\ \boxed{D_{yx}} & D_{yy} & \boxed{D_{yz}} \\ \boxed{D_{zx}} & \boxed{D_{zy}} & D_{zz} \end{pmatrix}$$

The Inverse Covariance Matrix

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \mathbf{\Lambda}^{-1} = \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$$

where $\lambda_i = \dfrac{1}{\sigma_i^2}$ and $\sigma_i^2 = 2D_i\tau$

eigenvalues of the inverse covariance matrix are

The meaning of the eigensystem

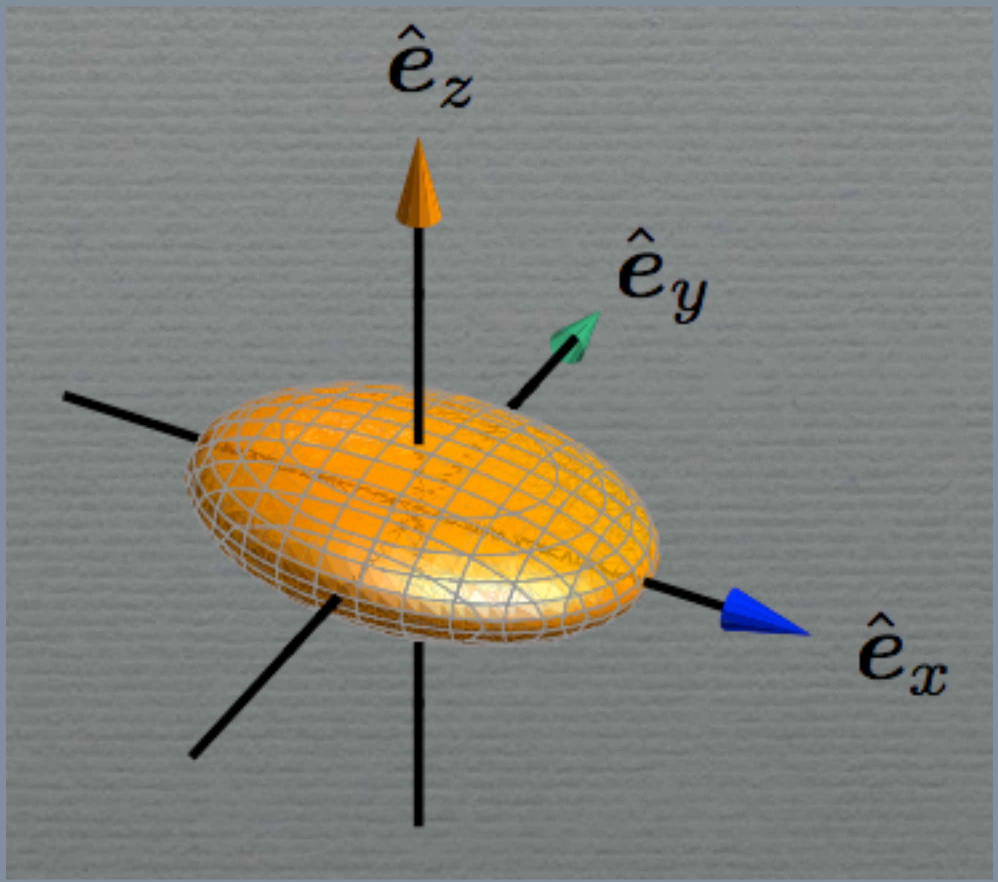
The eigenvectors of ***D*** are the unique directions along which the molecular displacements are uncorrelated

The eigenvalues $\{Dx, Dy\}$ are the principal diffusivities along these directions

The 3D Gaussian Distribution:

$$P(\boldsymbol{r}|\boldsymbol{r}_0, \tau) \sim N(\boldsymbol{r}_0, \boldsymbol{\Sigma})$$

$$\boldsymbol{r} = \{x, y, z\}$$



Covariance matrix

Diffusion Tensor

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma^2_{xx} & 0 & \\ 0 & \sigma^2_{yy} & 0 \\ 0 & 0 & \sigma^2_{zz} \end{pmatrix} = 6\tau \begin{pmatrix} D_x & 0 & \\ 0 & D_y & 0 \\ 0 & 0 & D_z \end{pmatrix}$$

The Diffusion Tensor

The three eigenvectors of ***D***

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$$

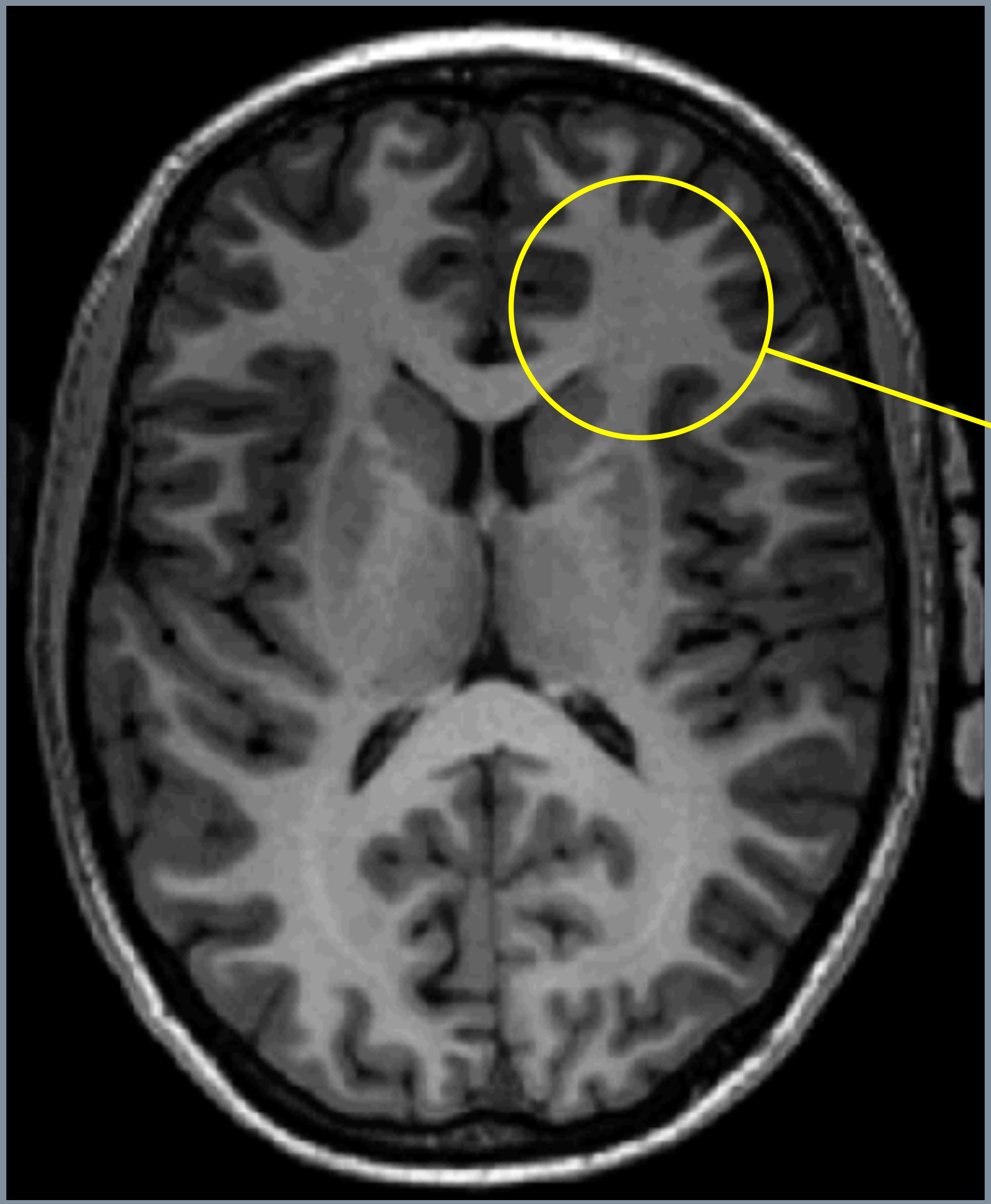
are the three unique directions along which the molecular displacements are uncorrelated

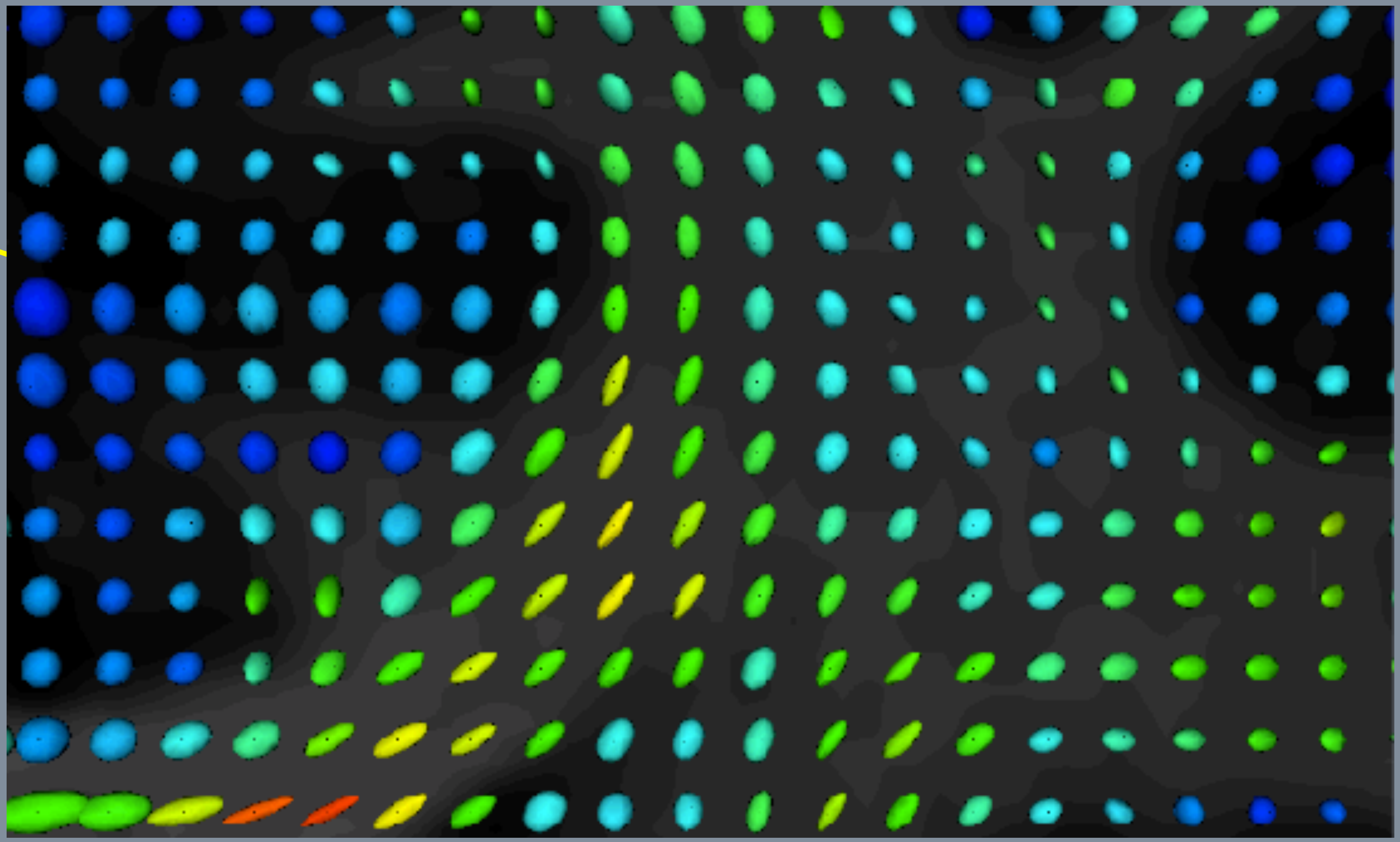
The three eigenvalues of ***D***

$$\{D_x, D_y, D_z\}$$

are the ADC values along these directions

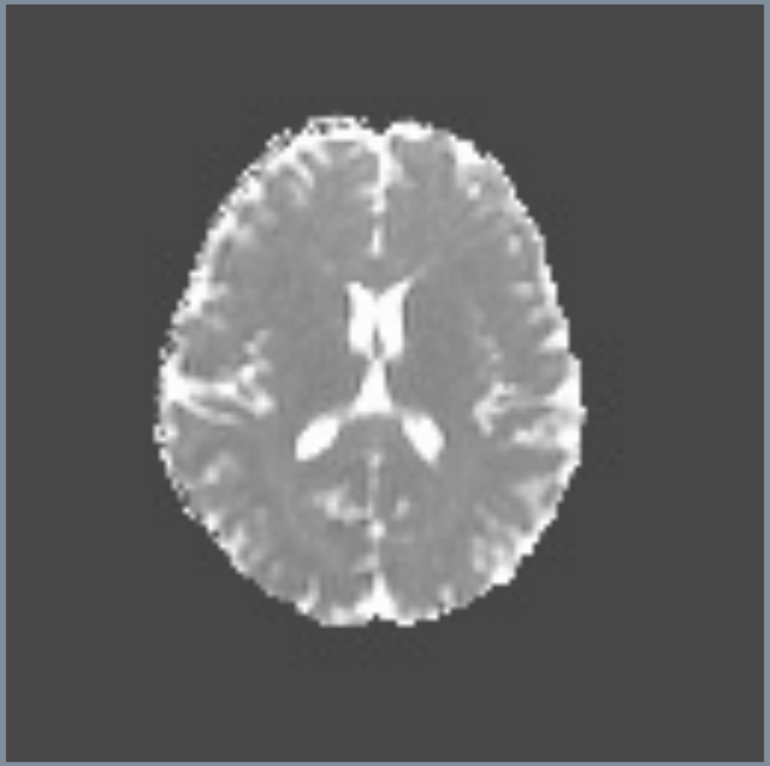
Reconstructed Diffusion Tensor is a VERY simplified model!



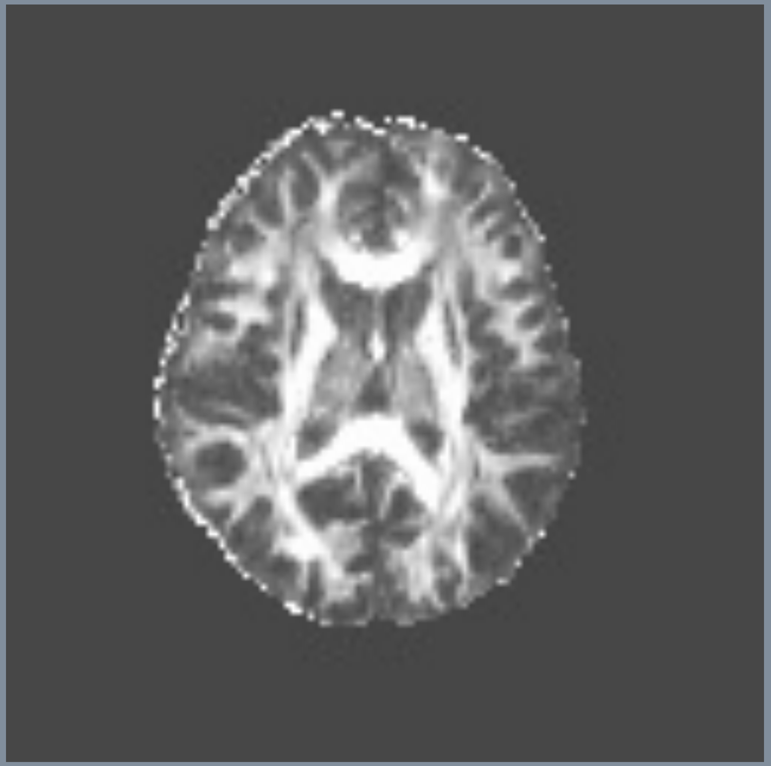


diffusion ellipsoids

Measured DTI Parameters are averaging over A LOT of physiology!



Mean diffusion



Diffusion anisotropy

Slides from previous lecture to add here

END