2 Coordinate systems

2.1 Motivation

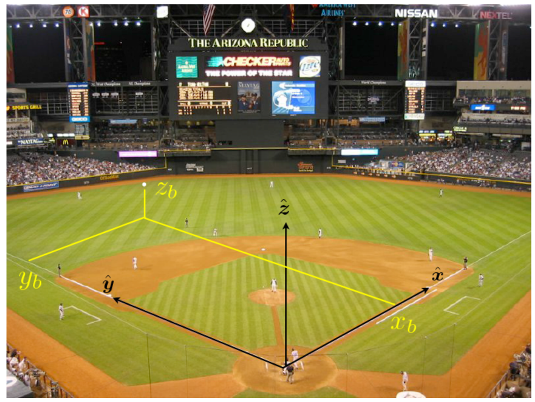
Coordinates systems play a crucial role in the way physical systems are described. Complicated problems can often be greatly simplified or be made more intuitively clear by a judicious choice of coordinate system. In addition, the way quantities are represented in these systems can also make a tremendous difference in the ease with which calculations are made and physical insight is obtained. We begin by describing the two coordinates systems familiar in everyday life from describing locations in 3-dimensional space: the Cartesian and the spherical coordinate system. It turns out that these are two of the most important coordinate systems in MRI, along with their 2-dimensional versions, the rectangular and polar coordinate systems.

2.2 Cartesian coordinates

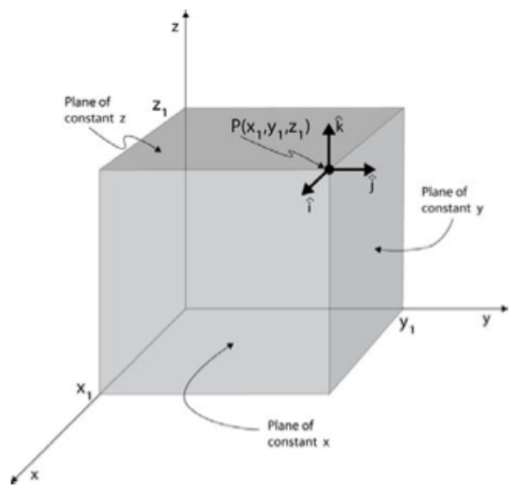
Specifying "where" something is located is something that we are constantly doing in everyday life. Often by "where" we mean "where in space". Consider, for example, the location of the baseball on the baseball field shown in Figure 2.1a. This is most easily described by defining three perpendicular axes along the first base line (x), the third base line (y), and the vertical through home plate (z) and the locations are described by the three parameters, three spatial coordinates, $\{x, y, z\}$. These three axes are said to form the *basis* for the coordinate system. The location of the intersection of the three axes (home plate) is called the *origin* and defined to be the point $\{0, 0, 0\}$ and is the point relative to which the other points are measured. This particular coordinate system, defined by three perpendicular axes that intersect at the origin is called the *Cartesian coordinate system*. An important quality of the Cartesian coordinate system is that the axes $\{x, y, z\}$ are perpendicular, or *orthogonal*, to one another. This has the important effect that if we move along one of the coordinate axes, our location as specified by the other coordinate axes remains unchanged. If the ball in Figure 2.1a were to move only along the y-axis for a distance Δy, its location would then be $\{x_b, y_b + \Delta y, z_b\}$. The values of the other two coordinates do not change. This independence can greatly simplify the description or analysis of a system that is changing as a function of the coordinates. A coordinate system with this property is called an *orthogonal* coordinate system. The utility of such a coordinate system is that the location $\{x_b, y_b, z_b\}$ of the baseball in 3D space can be described by the independent locations along each of these axes. This is shown in Figure 2.1. [1]

There is another more subtle but important concept besides the relative orientation of the axes that is needed to completely describe the Cartesian coordinate system, and that is the directions

[1] graphic from http://www4.wittenberg.edu/maxwell/CoordinateSystemReview.pdf



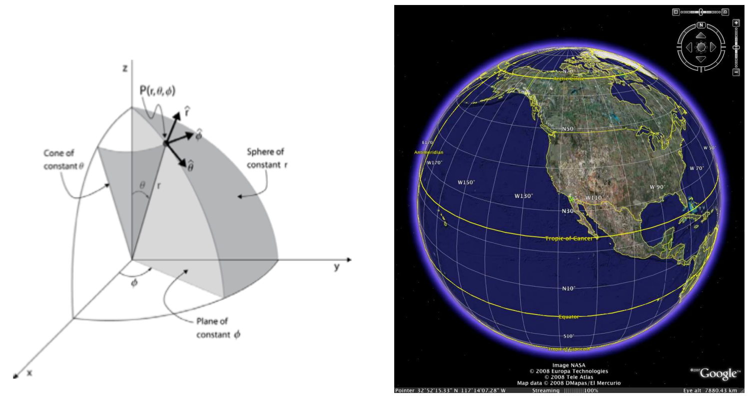
(a) Cartesian coordinate system: The location of a baseball in 3D space described by the independent Cartesian coordinates $\{x_b, y_b, z_b\}$.



(b) Representation of a 3D point p in a Cartesian coordinate system. Its location $\{x_1, y_1, z_1\}$ is described by the distance along each of the three perpendicular axes $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$.

Figure 2.1 The Cartesian coordinate system.

of the positive and negative directions. In Figure 2.1a the positive directions of $\{x, y, z\}$ are along first base, third base, and above home plate, respectively. But this is actually arbitrary. We could have defined the vertical as the $-z$ direction, for example. This distinction will make a difference when we discuss concepts such as rotations, however, so we need a way to describe and categorize this concept. The usual way to do this is to imagine an open right hand pointing along the positive x-axis. If you curl your fingers, they move towards the positive y-axis, and the thumb is pointing vertically. If this vertical axis is called the positive z-axis, then this coordinate system is called a *right-handed* coordinate system. If this vertical axis is called the positive $-z$-



(a) Parameters of the spherical coordinate system.

(b) The location of UCSD on the globe is most easily described in spherical coordinates.

Figure 2.2 Spherical coordinate system. The spherical system is composed of $\{r, \vartheta, \varphi\}$. Polar angle ϑ describes that angle of a vector from the the z-axis and azimuthal angle φ describes the angle in the $x - y$ plane from the x-axis. The radius r is the distance of the vector tip from the origin $\{0, 0, 0\}$, which is just the length of the vector.

axis, then this coordinate system is called a *left-handed* coordinate system (because if you have your left hand open along x and want to curl it along the same positive y direction, you'll need your thumb pointing down.)

In 2-dimensions, the Cartesian coordinate system is called the *rectangular* coordinate system. This system is useful for describing locations of objects in a plane laid out in a grid, such as a building on a city map, or pixels in an image. It is also the pattern of data samples collected in the majority of MRI acquisitions (Figure 2.4a).

2.3 Spherical coordinates

Another important coordinate system is the *spherical* coordinate system, which is familiar because we live on an approximately spherical object - the Earth (Figure 2.2b). This coordinate system is described by the three parameters $\{r, \vartheta, \varphi\}$, the radius, the polar angle, and the azimuthal angle, respectively (Figure 2.2a). These parameters are thus the *basis* for the spherical coordinate system. The lines are of constant ϕ are just the familiar lines of *latitude* and the the lines are of constant θ are the lines of *longitude*. This is shown in Figure 2.2. [2] The spherical coordinate system is also an orthogonal coordinate system. If we consider the Earth to be a perfect sphere, as we move on its surface we are at a constant radius (the distance from the center of the Earth). We can move along a line of constant longitude without changing the latitude, and vise versa. We can also rise above the surface of the earth in the radial direction at a particular point on the surface defined by a particular set of angle $\{\theta, \phi\}$.

The latitude and longitude lines on a globe describe the angle up from the equator and the angle

[2] graphic from http://www4.wittenberg.edu/maxwell/CoordinateSystemReview.pdf

away from the prime meridian, respectively. The equator divides the Earth into two hemispheres (northern and southern) about the origin of the latitude lines, while the prime meridian divides the Earth into two hemisphere (eastern and western) about the origin of the longitude lines. The prime meridian therefore defines the origin of the *azimuthal* angle φ and the origin of the elevation angle is the equator. Notice that instead of specifying the angle from the equator up to the North Pole, we could equivalently defined the angle ϑ, called the *polar angle* or *colatitude*, in the opposite direction, from the North Pole down to the equator. The elevation angle is then just $90° - \vartheta$. In the definitions of spherical coordinate systems it is common to use the polar angle. We can thus describe our three-dimensional space by the three coordinates $\{r, \vartheta, \varphi\}$, the radius, the azimuthal angle, and the polar angle, respectively.

2.4 The relationship between Cartesian and spherical coordinates

It is important to understand that the different coordinate systems can be just ways to describe the same space of parameters. We choose one over another because it makes the description easier, more intuitive, or economical. But if two coordinates systems that describe the same space (for example, the Cartesian and spherical descriptions of 3-dimensional space), there must be a way to mathematically describe one set of coordinate in terms of the others. Indeed, we can write the Cartesian coordinates $\{x, y, z\}$ in terms of the spherical coordinates $\{r, \vartheta, \varphi\}$:

$$x = r\sin\vartheta\cos\varphi \tag{2.1a}$$
$$y = r\sin\vartheta\sin\varphi \tag{2.1b}$$
$$z = r\cos\varphi \tag{2.1c}$$

And we can write the spherical coordinates in terms of the Cartesian coordinates as

$$r = \sqrt{x^2 + y^2 + z^2} \tag{2.2a}$$
$$\vartheta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \tag{2.2b}$$
$$\varphi = \arctan(y/x) \tag{2.2c}$$

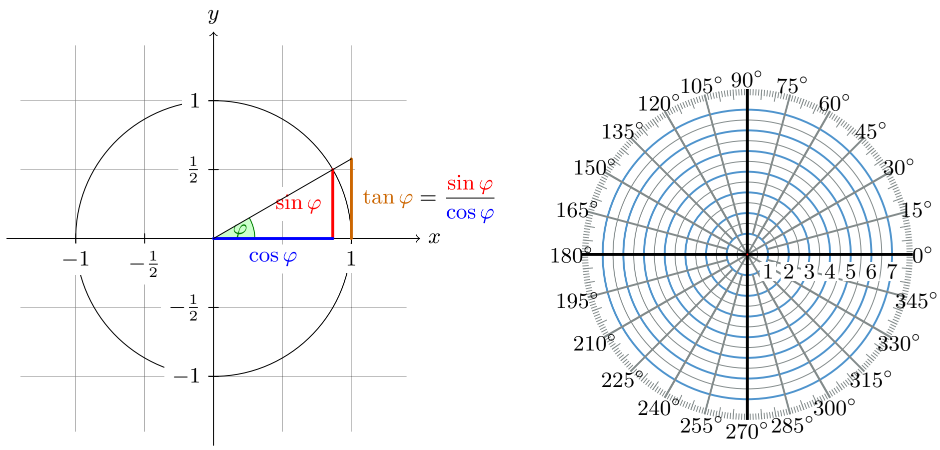
Such *coordinate transformations* will be discussed in greater detail in Section **??**.

2.5 Polar coordinates

The two dimensional (planar) version of the the Cartesian coordinate system is the rectangular coordinate system and the two dimensional version of the spherical coordinate system is the *polar* coordinate system. One can think of it as the coordinates in the spherical system if we just stay at the equator ($\vartheta = 90°$). With the φ origin chosen along the $+x$ direction, a typical representation of the polar coordinate system is shown graphically in Figure 2.3b where the angles are shown (arbitrarily) at 30° increments.

The rectangular coordinates in terms of the polar coordinates is

$$x = r\cos\varphi \tag{2.3a}$$
$$y = r\sin\varphi \tag{2.3b}$$



(a) Polar coordinates in terms of rectangular coordinates.

(b) Representation of the polar coordinate system shown at 30° increments, with the φ origin at the $+x$ axis.

Figure 2.3 The polar coordinate system

and the polar coordinates in terms of the rectangular coordinates is

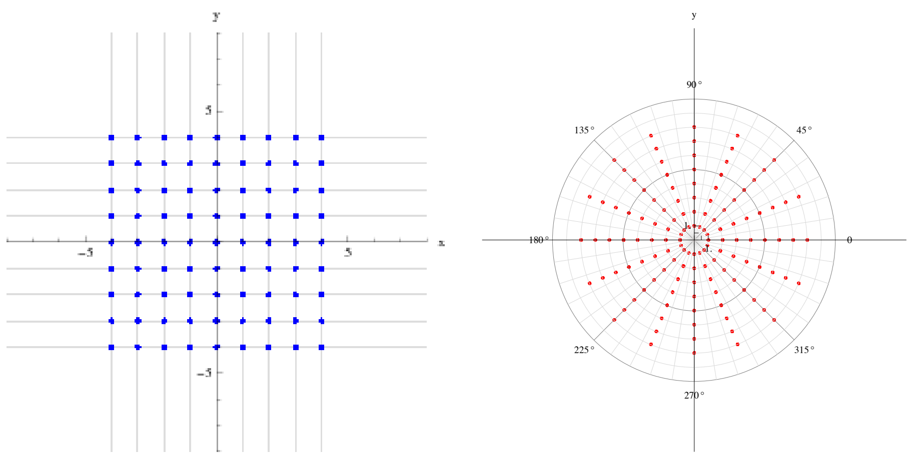
$$r = \sqrt{x^2 + y^2} \tag{2.4a}$$

$$\varphi = \arctan(y/x) \tag{2.4b}$$

The polar coordinate system is important when we discuss complex numbers because it is convenient for the representation of problems where magnitude and phase are important. Generally, polar coordinate systems are useful when a problem has some structure where there is a "center" from which a radius can be defined in two-dimensions. For example, as we will see in later chapters, the raw 2D MRI data that is collected from the scanner (*k-space* data) has an inherent cylindrical symmetry that is utilized by certain acquisition schemes. So, while most MRI data is collected on a Cartesian grid (Figure 2.4a), data may also be collected along the radial direction with different angular increments (Figure 2.4b). Such an acquisition scheme is thus naturally described in terms of polar coordinates.

2.6 Generalizations and higher dimensions

The coordinate system descriptions have focused on the familiar spatial dimensions but these concepts are actually very general. Imagine, for example, a classroom of students of different heights h, masses m, hair lengths l, and body temperatures T, all in different locations $\{x, y, z\}$, all moving around at a function of time t. This describes an eight dimensional system where all the coordinate $\{x, y, z, t, h, m, l, T\}$ are independent and can be desribed by an eight-dimensional Cartesian coordinate system. These generalization can be facilitated by more compact mathematical descriptions, as we will see in our discussion of vectors and tensors. (Or, perhaps they're not independent. Might hair length be related to temperature?)



(a) Cartesian grid. (b) Polar grid.

Figure 2.4 Sampling schemes can use different coordinate systems. Points on a Cartesian grid are shown in blue, and those on a (planar) sphereical grid are shown in red.

3 Vectors

3.1 Motivation

A great deal of the work of physics is trying to succinctly describe quantitative aspects of physical phenomena. This is greatly facilitated by mathematical constructs that allow concise physical descriptions. One such construct is the *vector* which allows the concatenation of multiple (in fact, an infinite number) of parameters into a single object which has well defined mathematical properties, and the additional benefit of illuminating geometrical interpretations.

3.2 Scalars

A *scalar* is a just a real number. Scalars satisfy the properties of being *associative* for addition and multiplication

$$(a+b)+c = a+(b+c) \tag{3.1a}$$
$$(ab)c = a(bc) \tag{3.1b}$$

for scalars a, b, c and *commutative* for addition and multiplication

$$a+b = b+a \tag{3.2a}$$
$$ab = ba \tag{3.2b}$$

It is useful to think of a scalar as a number that *scales* the length of a vector, as we shall see.

A useful shorthand notation that has been invented for describing that a number a is a real number is $a \in \mathbb{R}$ where $\mathbb{R}$ stands for the real numbers and $\in$ mean "is an element of". There is another notation that is useful in defining where a number lies relative to numbers. There are really just four options: Either x lies up to and includes or does not include a and/or b at its limits. This notation is the use a *parenthesis* "(" or ")" to denote that x does *not* include the limiting values, or *square brackets* "[" or "]" to denote that it does. The four choices are thus

$$(a,b) \equiv a < x < b \tag{3.3a}$$
$$[a,b) \equiv a \leq x < b \tag{3.3b}$$
$$(a,b] \equiv a < x \leq b \tag{3.3c}$$
$$[a,b] \equiv a \leq x \leq b \tag{3.3d}$$

This notation is just a convenient shorthand.

3.3 Simple vectors

Consider again the location of the baseball in Figure 3.1 relative to the origin (home plate) which we can visualize by drawing an arrow from home plate to the baseball. This arrow has a length (or magnitude) and a direction and is described by a mathematical construct called a *vector*. The initial point of this vector is the origin $\{0, 0, 0\}$ and the final points is the location of the baseball in terms of the *scalars* (real numbers) $\{x_b, y_b, z_b\}$. Vectors are implicitly assumed to have their tail (starting point) at the origin $\{0, 0, 0\}$ and are identified by their end points in the form of a *column* vector. The baseball the location in the Cartesian axes is represented by the vector $\boldsymbol{v}$ (vectors are typically signified by boldface type) represented by the notation:

$$\boldsymbol{v} \equiv \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} \tag{3.4}$$

where the vector *components* $\{v_x, v_y, v_z\}$ the distance along each of the separate (basis) axes $(\{x, y, z\})$ the ball is located. Two equivalent, but slight more general, ways of writing Eqn 3.4 use the notation of the vector with subscripts to denote the component of the vector along the basis axes $\{x_b, y_b, z_b\} = \{v_x, v_y, v_z\}$ or by the component number $\{x, y, z\} = \{1, 2, 3\}$. Hence,

$$\boldsymbol{v} \equiv \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{3.5}$$

The latter form is particularly useful when we generalize vectors to higher dimensionals. For example, a set of n data points $\{d_1, d_2, \ldots, d_n\}$ from a single voxel time course in an FMRI experiment can be represented by an n-dimensional vector.

$$\boldsymbol{d} \equiv \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \tag{3.6}$$

The "space" is now not as obvious as 3-dimensional space–it's the n-dimensional "signal space"– but the mathematical representation remains the same.

A useful shorthand notation that has been invented for describing that a vector $\boldsymbol{v}$ is composed of real numbers and of dimension n is $\boldsymbol{v} \in \mathbb{R}^n$.

3.4 The vector transpose

Interchanging the rows and the columns of a vector $\boldsymbol{v}$ is called the *transpose* of $\boldsymbol{v}$, denoted $\boldsymbol{v}^t$. For example, the transpose of Eqn 3.6 is

$$\boldsymbol{d}^t \equiv (d_1, d_2, \ldots, d_n) \tag{3.7}$$

The transpose of a column vector is a *row* vector, and vise versa. A column vector of length n has n rows, whereas a row vector of length n has n columns. Whether one uses a column vector or a row vector to represent a set of numbers of often arbitrary and a matter of convenience

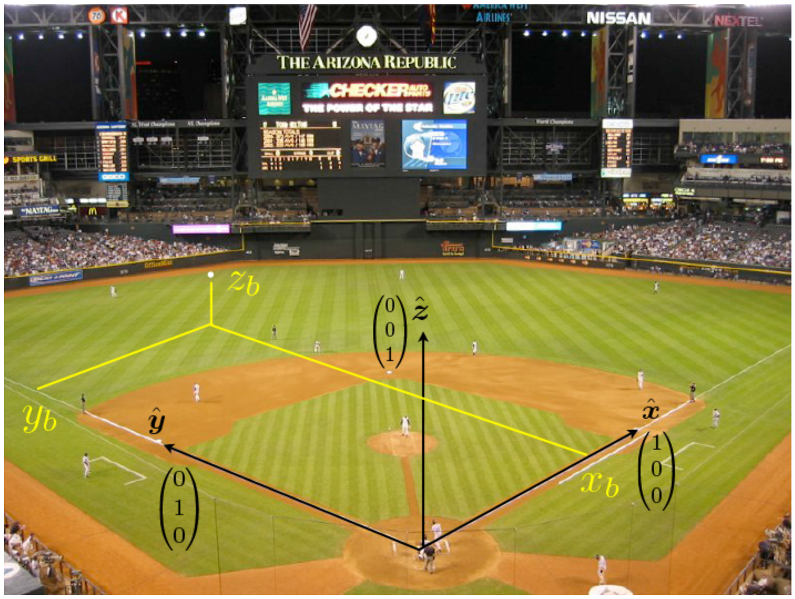


Figure 3.1 The Cartesian basis vectors.

or convention. What is not arbitrary is the relationship between the two, and the mathematical rules for their manipulation. A more general way to think about the transpose is that the columns are exchanged with the rows.

3.5 Basis vectors

There are actually three other vectors shown in Figure 3.1 which are along the basis axes and of equal length (the distance from home plate to first base). If we define this length to be the unit of length in this "space" (the stadium) then all of these are vectors of length 1 - and are called *unit* vectors, and denoted by a hat symbol ˆ. These unit vectors pointing along the basis axes are called *basis* vectors, which are typically written with the symbol $\boldsymbol{e}$ and a subscript to denote the basis component. Thus our baseball field basis vectors are:

$$\boldsymbol{e}_x = \hat{\boldsymbol{x}} \equiv \begin{pmatrix}1\\0\\0\end{pmatrix} \quad , \quad \boldsymbol{e}_y = \hat{\boldsymbol{y}} \equiv \begin{pmatrix}0\\1\\0\end{pmatrix} \quad , \quad \boldsymbol{e}_z = \hat{\boldsymbol{z}} \equiv \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{3.8}$$

and are shown in Figure 3.1. Be cognisant of the fact that basis vectors are always unit vectors, but not all unit vectors are basis vectors, of course.

Similarly, for the spherical coordinate system of Section 2.3 there are basis vectors along the basis axes $\boldsymbol{e}_r = \hat{\boldsymbol{r}}, \boldsymbol{e}_\theta = \hat{\boldsymbol{\theta}}, \boldsymbol{e}_\phi = \hat{\boldsymbol{\phi}}$. These are shown in Figure 2.2a.

One comment on notation. In situations involving well-known coordinate systems such as the Cartesian and spherical coordinate systems, it is common to express the basis vectors in terms of the coordinates they describe, and with a hat, such as $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ and $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$, rather than in the form $\boldsymbol{e}_x$, etc, because it is more intuitive and gives less cluttered equations.

3.6 Vector addition, subtraction, and scalar-vector multiplication

Two basic properties of vectors is that multiplication of a vector by a scalar is equivalent to multiplying each component of that vector by that scalar. For example, multiplying the vector Eqn 3.5 by the scalar a gives

$$a\boldsymbol{v} = \begin{pmatrix} a\,v_1 \\ a\,v_2 \\ a\,v_3 \end{pmatrix} \tag{3.9}$$

Consider a two-dimensional in the Cartesian basis with scalar coefficient v_x and v_y multiplied by a scalar a:

$$a\boldsymbol{v} = av_x\hat{\boldsymbol{x}} + av_y\hat{\boldsymbol{y}} \tag{3.10}$$

Each coefficient is just scaled (e.g., stretched or shrunk) by the a as shown in Figure 3.2a for $v_x = v_y = 1$.

When two vectors are added (or subtracted), this is done by adding (or subtracting) their individual components:

$$\boldsymbol{u} + \boldsymbol{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix} \quad , \quad \boldsymbol{u} - \boldsymbol{v} = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{pmatrix} \tag{3.11}$$

From these two simple facts, we see that the vector Eqn 3.4 representing the baseball's location can be written:

$$\boldsymbol{v} = x_b\hat{\boldsymbol{x}} + y_b\hat{\boldsymbol{y}} + z_b\hat{\boldsymbol{z}} \tag{3.12}$$

Example 3.1 Show Eqn 3.12

Solution

From Eqn 3.9 in the first step

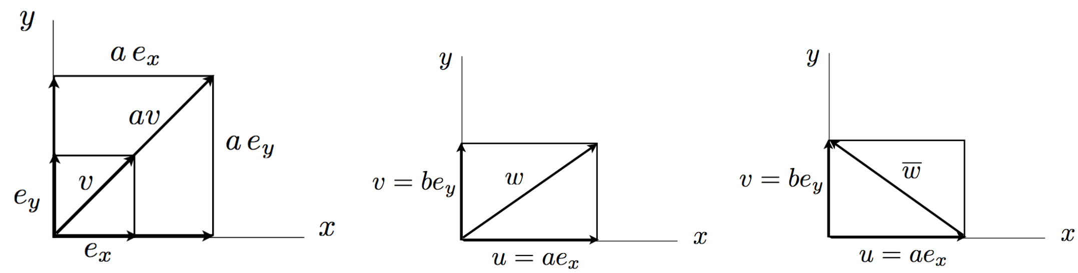
$$\boldsymbol{v} = x_b \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y_b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z_b \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_b \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y_b \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z_b \end{pmatrix} = \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

and using Eqn 3.11 in the second step.

Consider two vectors in the three-dimensional Cartesian basis:

$$\boldsymbol{u} = u_x\hat{\boldsymbol{x}} + u_y\hat{\boldsymbol{y}} + u_z\hat{\boldsymbol{z}} \tag{3.13a}$$

$$\boldsymbol{v} = v_x\hat{\boldsymbol{x}} + v_y\hat{\boldsymbol{y}} + v_z\hat{\boldsymbol{z}} \tag{3.13b}$$



(a) Vector scalar multiplication: $a\boldsymbol{v} = a\hat{\boldsymbol{x}} + a\hat{\boldsymbol{y}}$

(b) Vector addition: $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v} = a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}}$

(c) Vector subtraction: $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v} = a\hat{\boldsymbol{x}} - b\hat{\boldsymbol{y}}$

Figure 3.2 Vector math for some simple vectors. (**change to $\hat{\boldsymbol{x}}$, etc!**)

Adding and subtracting these vectors gives

$$\boldsymbol{u} + \boldsymbol{v} = (u_x + v_x)\hat{\boldsymbol{x}} + (u_y + v_y)\hat{\boldsymbol{y}} + (u_z + v_z)\hat{\boldsymbol{z}} \tag{3.14}$$

$$\boldsymbol{u} - \boldsymbol{v} = (u_x - v_x)\hat{\boldsymbol{x}} + (u_y - v_y)\hat{\boldsymbol{y}} + (u_z - v_z)\hat{\boldsymbol{z}} \tag{3.15}$$

The coefficients along the basis vectors just add or subtract. This is shown in Figure 3.2b and Figure 3.2c. Geometrically, adding two corresponds to putting the base of the second vector at the tip of the first vector, and drawing a line from the base of the first vector to the tip of the second vector.

3.7 The magnitude and angle of a vector

The vector *magnitude*, or *length*, is the distance from its base to its tip, and denoted $|\boldsymbol{v}|$. In the Cartesian basis, this is just the generalization of the familiar Pythagorean Theorem, the square root of the sum of the squares of the components v_i along each axis $\hat{\boldsymbol{e}}_i$:

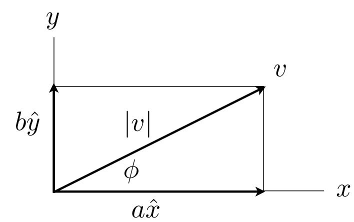
$$|\boldsymbol{v}| = \sqrt{\sum_{i=1}^{n} v_i^2} \tag{3.16}$$

For the simple examples in Cartesian coordinates of a two dimensional vector in Figure 3.3a $\boldsymbol{v} = a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}}$, the magnitude is $|\boldsymbol{v}| = \sqrt{a^2 + b^2}$ and for the three-dimensional vector in Figure 3.3b $\boldsymbol{v} = a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}} + c\hat{\boldsymbol{z}}$, the magnitude is $|\boldsymbol{v}| = \sqrt{a^2 + b^2 + c^2}$ The simplicity of expression Eqn 3.16 is a result of the fact that the Cartesian coordinate system in an orthogonal coordinate system, so that the components of the vector can be described by the *independent* components along the separate (and orthogonal) axes.

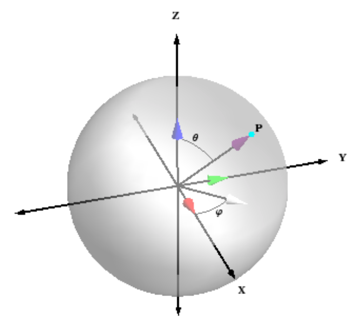
The *angle*, or *direction*, of a vector must be defined relative to some axis, which is usually chosen by convention. For the simple example of a two dimensional vector in Cartesian coordinates Figure 3.3a $v = a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}}$, the angle is typically measured relative to the x-axis, and is given from simply trigonometry to be

$$\phi = \tan^{-1}(b/a) \tag{3.17}$$

where $\tan^{-1}$ is the arctangent function. For the three-dimensional vector in Figure 3.3b, two



(a) A two-dimensional vector in Cartesian coordinates. The magnitude is $|\boldsymbol{v}| = \sqrt{a^2 + b^2}$ and angle is $\theta = \tan^{-1}(b/a)$.



(b) A three-dimensional vector in Cartesian coordinates. The magnitude is $|\boldsymbol{v}| = \sqrt{a^2 + b^2 + c^2}$, the azimuthal angle is $\phi = \tan^{-1}(b/a)$, and the polar angle is $\theta = \tan^{-1}(c/a)$. (**fix figure!**)

Figure 3.3 Vector magnitude and angle.

angles can be used to define the orientation of the vector: the angle from the $\hat{\boldsymbol{z}}$ director, called the *polar angle*, and the angle in the $x - y$-plane from the $\hat{\boldsymbol{x}}$ axis, called the *azimuthal angle*. These angles are given by

$$\theta = \tan^{-1}(c/a) \tag{3.18a}$$

$$\phi = \tan^{-1}(b/a) \tag{3.18b}$$

The notation $\{\theta, \phi\}$ for the polar and azimuthal angle, respectively, is the standard in the physics literature. In the mathematical literature, these are often swapped, so it is prudent to pay attention to the definitions in each context.

3.8 Interlude: The Unit Vector

A very useful vector is the vector of unit length, called a *unit vector*. In fact, we have already used unit vectors in Section 3.5 to define our coordinate axes. Unit vectors are very useful for this purpose because, for example, in an orthogonal coordinate system such as the Cartesian system, the component of a vector along any direction can be expressed as some scalar times the unit vector along that direction, as we saw in Section 3.6. Now that we have defined the vector magnitude, it is clear that we can take any vector $\boldsymbol{v}$, divide it (or *normalize* it) by its magnitude $|\boldsymbol{v}|$, to produce a unit vector $\hat{\boldsymbol{v}}$ in the diretion of $\boldsymbol{v}$:

$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \tag{3.19}$$

The hat symbol is the standard notation for a unit vector.

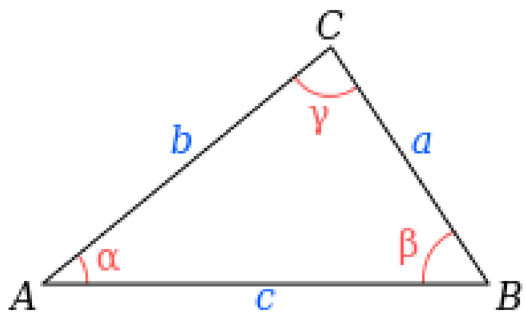


Figure 3.4 The Law of Cosines.

3.9 The angle between two vectors: The Law of Cosines

The *Law of Cosines* states that the length of the vector $\boldsymbol{c}$ is related to the length of the other two sides $\boldsymbol{a}$ and $\boldsymbol{b}$, and the angle θ between them by[1]

$$|\boldsymbol{c}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\,|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\gamma \tag{3.20}$$

This is shown in Figure 3.4.

3.10 Multiplication of vectors: The Dot or Inner Product

There are actually several rules for multiplying together two vectors. The simplest, and most ubiquitous, is the rule that the rows of one are multiplied by the columns of the other, and the results are added together. This only has meaning if the number of rows in the first is equal to the number of columns in the second. This is called the *dot product* or *inner product* of two vectors, and is symbolized by a dot between the two vectors: $\boldsymbol{u}\cdot\boldsymbol{v}$. Consider two three-component vectors, the row vector $\boldsymbol{u} = \{u_1, u_2, u_3\}$ and the column vector $\boldsymbol{v} = \{v_1, v_2, v_3\}^t$. The number of columns in $\boldsymbol{u}$ is equal to the number of rows in $\boldsymbol{v}$ and so we can form the dot product between the two:

$$\boldsymbol{u}\cdot\boldsymbol{v} \equiv \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{j=1}^{n} u_j v_j \tag{3.21}$$

In Section 3.7 the vector magnitude as found by summing the squares of the components along the separate axes, adding them, then taking the square root (Eqn 3.16). A compact way of writing this is to notice that if we set $\boldsymbol{u} = \boldsymbol{v}^t$ in Eqn 3.21 then we can rewrite the magnitude of $\boldsymbol{v}$ (Eqn 3.16) in the form

$$|\boldsymbol{v}| = \sqrt{\boldsymbol{v}^t \cdot \boldsymbol{v}} \tag{3.22}$$

One very common but potentially confusing shorthand notation is the elimination of the dot when writing the dot product so that $\boldsymbol{v}^t \cdot \boldsymbol{v}$ is written $\boldsymbol{v}^t\boldsymbol{v}$.

[1] I think this equation is getting closer than Eqn **??** was, but I still don't think there should be a minus sign.

We get an interesting and useful result if we consider the magnitude of the difference between two vectors $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$:

$$|\boldsymbol{w}|^2 = |\boldsymbol{u} - \boldsymbol{v}|^2 = |\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 - 2\boldsymbol{u} \cdot \boldsymbol{v} \tag{3.23}$$

This must be the same as the Law of Cosines (Eqn 3.20) from which we can conclude that another form of the dot product is

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}|\,|\boldsymbol{v}| \cos\theta \tag{3.24}$$

where θ is the angle between the two vectors.

An illuminating example is the dot product between the basis vectors Eqn 3.8 for the Cartesian basis:

$$\hat{\boldsymbol{x}}^t \cdot \hat{\boldsymbol{y}} = \hat{\boldsymbol{y}}^t \cdot \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}^t \cdot \hat{\boldsymbol{x}} = 0 \tag{3.25}$$

which, from Eqn 3.24, means that the angle between any of these vector is 90°. In other words, they are *orthogonal.* This we already knew from Figure 3.1 but now we have a concise mathematical statement of that fact. In fact, a vanishing dot product between two vectors is synonymous with saying they are orthogonal. Also note the important fact that the order of the vectors does not matter in the dot product: $x^t \cdot y^t = y^t \cdot x^t$.

The dot product is maximized when $\cos\theta = 1$, or $\theta = 0$, i.e., when the vectors are collinear. It vanishes then the $\cos\theta = 0$, or $\theta = 90°$, i.e., when the vectors are perpendicular.

Problems

3.1 Show that the basis vectors $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ in the spherical coordinate systems are orthogonal, i.e., that $\hat{\boldsymbol{r}}^t \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}^t \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}^t \cdot \hat{\boldsymbol{r}} = 0$

3.11 Geometry of the dot product: Orthogonal Projections

The equation Eqn 3.24 can be rearranged into the form

$$\boldsymbol{u} \cdot \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \boldsymbol{u} \cdot \hat{\boldsymbol{v}} = |\boldsymbol{u}| \cos\theta \tag{3.26}$$

which says that the dot product of $\boldsymbol{u}$ with $\hat{\boldsymbol{v}}$ is just the component of $\boldsymbol{u}$ along $\hat{\boldsymbol{v}}$, as shown in Figure 3.5. This is called the *orthogonal projection* of $\boldsymbol{u}$ onto $\hat{\boldsymbol{v}}$ because the difference vector $\boldsymbol{u} - \boldsymbol{u} \cdot \hat{\boldsymbol{v}}$ (the dotted line in Figure 3.5, called the *orthogonal complement*) is always orthogonal to $\boldsymbol{v}$. Thus the dot product can be viewed as the orthogon projection of $\boldsymbol{u}$ onto $\hat{\boldsymbol{v}}$, scaled by the magnitude of $\boldsymbol{v}$. Note that since the dot product is commutative for vectors, we can exchange $\boldsymbol{u}$ with $\boldsymbol{v}$ for the above discussion. The geometry of the dot product is a very useful concept in many applications, such as in data analysis where the estimation of model parameters often involves the dot product of the data vector with the model vector. In this case, the signal "explained" by the data involved the orthogonal projection of the data onto the model, and the error is the orthogonal complement.

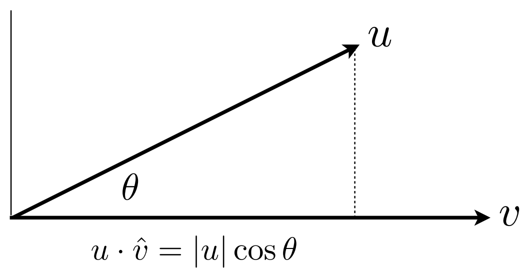


Figure 3.5 The orthogonal projection.

3.12 Interlude: The Dot Product and Equations

Vector notation and the dot product provide a very convenient and concise way to write equations. Consider the simple equation of a line:

$$y = \alpha + \beta x \tag{3.27}$$

This can be written in a more compact form by defining the vectors

$$\boldsymbol{\alpha} \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \boldsymbol{f} \equiv \begin{pmatrix} 1 \\ x \end{pmatrix} \tag{3.28}$$

The line can then be written dot product between these two vectors

$$y = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \alpha \; 1 + \beta \; x \tag{3.29}$$

where components of like colors are multiplied together. That is

$$y = \boldsymbol{\alpha}^t \boldsymbol{f} \tag{3.30}$$

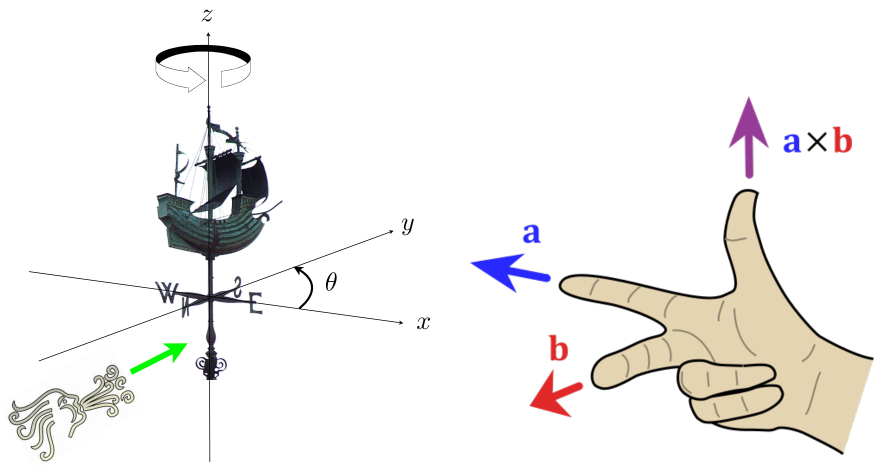
(where we've purposefully eliminated the dot to start adjusting you to the aforementioned convention) or, in component form,

$$y = \sum_{i=1}^{2} \alpha_i f_i \tag{3.31}$$

These representations (Eqns 3.30- 3.31) become a powerful notation when we work with equations of very large dimensions where writing out the individual terms becomes cumbersome and often not particularly illuminating, whereas the dot product representation of the equation does not change.

3.13 The Cross Product

Another important way to multiply two vectors is illustrated by the physical example in Figure 3.6a where a weathervan is oriented along the $\hat{\boldsymbol{z}}$ direction such that the East is oriented along $\hat{\boldsymbol{x}}$ and the South is oriented along $\hat{\boldsymbol{y}}$. If there is a wind from the North that pushes on the East



(a) A weathervane. (b) The right hand rule.

Figure 3.6 The vector cross product.

side, the weathervane rotates about $\hat{\boldsymbol{z}}$ by angle angle θ towards the South ($\hat{\boldsymbol{y}}$) axis. The vector product that captures the fact that rotating a vector along $\hat{\boldsymbol{x}}$ toward one along $\hat{\boldsymbol{y}}$ rotates the weathervane about the $\hat{\boldsymbol{z}}$ is the *cross product*.

The definition of cross product between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v} = |\boldsymbol{u}|\,|\boldsymbol{v}| \sin\theta\, \hat{\boldsymbol{w}} \tag{3.32}$$

The result of the cross product is another vector that points in direction orthogonal to both the vectors that form it. The magnitude of the resulting vector is maximized when $\sin\theta = 1$, or $\theta = 90°$, i.e., when the vectors are perpendicular. It vanishes then the $\sin\theta = 0$, or $\theta = 0°$, i.e., when the vectors are collinear.

One very important feature to remember about the cross product is that it must be defined within a particular coordinate system. This is most easily demonstrated by looking at the cross product of the unit vectors that define a coordinate system. Consider the unit vectors $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ that define Cartesian the coordinate system shown in Figure 3.6a. They are orthogonal, so $\sin\theta = 1$ in Eqn 3.32 and thus $\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}$. But if the coordinate system was defined such that $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ were swapped, this would be equivalent to reversing the order of this product in the standard Cartesian coordinate system to $\hat{\boldsymbol{y}} \times \hat{\boldsymbol{x}} = -\hat{\boldsymbol{z}}$ and the weathervane turns in the opposite direction. A nice pnemonic for remembering this is called the *right hand rule*, shown in Figure 3.6b. Pointing your index finger along $\hat{\boldsymbol{x}}$ and your middle finger along $\hat{\boldsymbol{y}}$ will result in your thumb being pointed along $\hat{\boldsymbol{z}}$. Reversing the order will result in your thumb pointing along $-\hat{\boldsymbol{z}}$. Because changing the order of the multiplication changes the sign of the result, the cross product is said to be *anti-commutative*.

Listing the cross products of all three of the Cartesian basis vectors allows us to introduce another useful notational convenience:

$$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \quad , \quad \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{x}} \quad , \quad \hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \tag{3.33a}$$

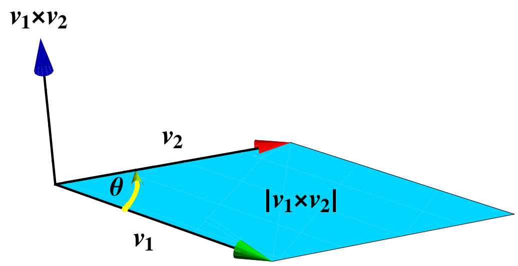


Figure 3.7 The geometry of the vector cross product. The cross product is a vector whos magnitude is the arear of the parallelogram defined by the two vectors, and whose direction is perpendicular to the plane defined by the two vectors. (**change vectors to $\boldsymbol{u}$ and $\boldsymbol{v}$**).

Similarly,

$$\hat{\boldsymbol{y}} \times \hat{\boldsymbol{x}} = -\hat{\boldsymbol{z}} \quad , \quad \hat{\boldsymbol{z}} \times \hat{\boldsymbol{y}} = -\hat{\boldsymbol{x}} \quad , \quad \hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}} = -\hat{\boldsymbol{y}} \tag{3.34a}$$

These sets of equations is often more compactly stated by noting that successive equations can be obtained with the substitution $x \to y, y \to z, z \to x$ at successive steps. Each element is substituted for the next element, until it hits the last element, in which case is cycles to the front of the line, so to speak. This is called *cyclic permutation*.

The components of the cross product $\boldsymbol{c} = \boldsymbol{u} \times \boldsymbol{v}$ can also be written in a useful way as

$$c_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} u_j v_k \tag{3.35}$$

where

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{cyclic permutations (i,j,k unique)} \\ -1 & \text{anti-cyclic permutations (i,j,k unique)} \\ 0 & \text{i,j,k not unique} \end{cases} \tag{3.36}$$

is called the *totally anti-symmetric Levi-Civita symbol*. It is only non-zero if the i, j, k are unique: $i \neq j \neq k$. A *cyclic* permutation is one in which each index is changed into the following index: i.e., $i \to j$, $j \to k$, $k \to i$. An *anti-cyclic* permutation is one in which each index is changed into the preceding index: i.e., $i \to k$, $j \to i$, $k \to j$. For example, $\varepsilon_{231} = 1$, $\varepsilon_{213} = 0$, $\varepsilon_{131} = 0$. We will encounter the form Eqn 3.35 repeatedly in the theory of rotational transformations in Section **??**.

3.14 Geometry of the cross product: parallelogram areas

There is a nice geometrical interpretation of the cross that is shown in Figure 3.7. The cross product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is a vector whose magnitude is the area of the parallelogram defined by $\boldsymbol{u}$ and $\boldsymbol{v}$ and whose direction is perpendicular to the plane defined by $\boldsymbol{u}$ and $\boldsymbol{v}$. The geometrical interpretation of the cross product magnitude as an area rarely comes up in

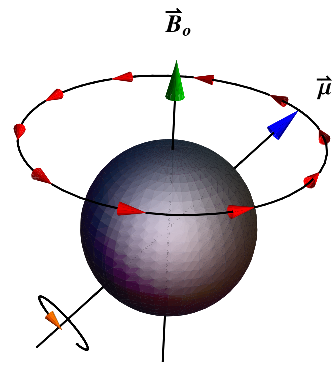


Figure 3.8 A nuclear spin with magnetic moment $\boldsymbol{\mu}$ in a static magnetic field $\boldsymbol{B}_0$ experiences a torque $\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B}_0$ causing precession of $\boldsymbol{\mu}$ about $\boldsymbol{B}_0$.

its application to MRI. Much more important is the geometrical insight of the direction of the resultant vector and its application to .
[2]

3.15 Interlude: The Cross Product, Spin, and Torque

Two very important physical quantities that are ubiquitous in MRI and involve the cross product are the *angular momentum* and *torque*. In classical mechanics, a particle of mass m rotating at a fixed radius $\boldsymbol{r}$ about the origin with velocity $\boldsymbol{v}$ has angular momentum

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \tag{3.37}$$

where the *linear momentum* is $\boldsymbol{p} = m\boldsymbol{v}$. The *torque* is the time rate of change of the angular momentum:

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F} \tag{3.38}$$

where $\boldsymbol{r}$ is radius and the force is $\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = m\frac{d\boldsymbol{v}}{dt} = ma$ where a is the acceleration.

The torque equation is central to the process of magnetic resonance because a nuclear spin (e.g., a proton in a water molecule) with magnetic moment $\boldsymbol{\mu}$ in a magnetic field $\boldsymbol{B}$ experiences a torque $\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B}$. For example, in a static magnetic field $\boldsymbol{B}_0$, the torque $\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B}_0$ causes precession of $\boldsymbol{\mu}$ about $\boldsymbol{B}_0$, as shown in Figure 3.8. This will be discussed in greater detail in Chapter 13.

Suggested reading

[2] Should we mention the vector triple product?

4 Complex numbers

4.1 Real, Imaginary, and Complex numbers

One of the properties we know about real numbers is that their square must be greater than or equal to zero (*non-negative*). So if the number a is real (i.e., $a \in \mathbb{R}$) then $a^2 \geq 0$. An *imaginary number*, on the other hand, has a squared value that is *less than* or equal to zero. To construct such a number it was necessary to invent[1] the quantity $i = \sqrt{-1}$. Multiplying a real number a by i then gives an imaging number $z = ia$ whose squared value is $z * z = (i * i) * z^2 = -z^2$. Complex numbers are a mathematical construct that facilitate a simple and compact method for describing rotations. A *complex number* is just the sum of a real and an imaginary number. So from two real numbers $\{a, b\} \in \mathbb{R}$ the complex number $z = a + ib$ can be constructed where a and b are called the *real* and *imaginary* components of z. Complex numbers are denoted by the symbol $\mathbb{C}$ so $z \in \mathbb{C}$.

The rule for adding two numbers is that the real parts are added together and the imaginary components are added together. Thus for two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, their sum is

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2) \tag{4.1}$$

Multiplication gives

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2) \tag{4.2}$$

where note the minus sign arises from $i * i = -1$.

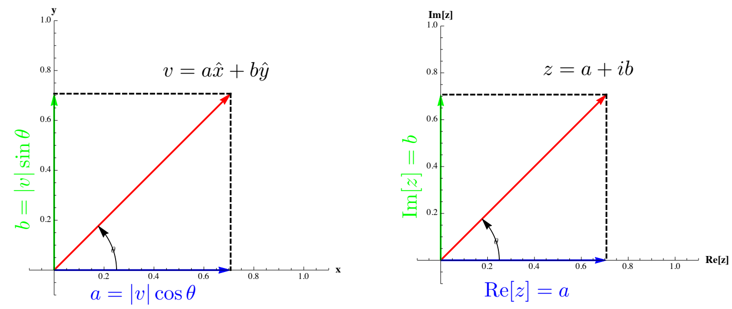
The representation of a two-dimensional vector $\boldsymbol{v} = a\hat{x} + b\hat{y}$ in the Cartesian $(x - y)$ plane Figure 4.1a can also be displayed in terms of the complex vector $z = a + ib$ if the x axis is associated with $\Re[z] = a$, where $\Re[]$ means the the real component of z, and the y axis is associated with $\Im[z] = v$, were $\Im[]$ means the imaginary component of z. This is called the *complex plane*, and shown in Figure 4.1b.

4.2 The magnitude and phase

Figure 4.1a and Figure 4.1b are two representations of the same vector $\boldsymbol{v}$, and so both must produce the same length $\|\boldsymbol{v}\|$ of the vector. For the Cartesian representation we already from Chapter 3 that this is $\|\boldsymbol{v}\| = \sqrt{a^2 + b^2}$. This requires that the definition of the *magnitude* of a complex vector be:

$$\|\boldsymbol{z}\| = \sqrt{z^* z} \tag{4.3}$$

[1] Invented by the Italian mathematician Rafael Bombelli (1526–1572).



(a) A vector in the $x - y$ (Cartesian) plane.

(b) A vector in the complex plane.

Figure 4.1 The representation of a vector in the Cartesian $(x - y)$ and complex planes.

where

$$z^* = a - ib \tag{4.4}$$

is called the *complex conjugate* of z. With this definition

$$z^* z = (a - ib)(a + ib) = a^2 + b^2 \tag{4.5}$$

so that $\|\boldsymbol{z}\| = \sqrt{a^2 + b^2}$ as required. The length of z is also called the *magnitude* of z.

Inspection of Figure 4.1b suggests another representation of complex vectors. The vector is characterized by two parameters: the vector magnitude $\|\boldsymbol{v}\|$ and the angle θ, also called the *phase*, which can be expressed by writing out the components as

$$v = \|\boldsymbol{v}\| \cos\theta \hat{\boldsymbol{r}} + \|\boldsymbol{v}\| \sin\theta \hat{\boldsymbol{i}} \tag{4.6}$$

where $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{i}}$ are the unit vectors in the real and imaginary axes and

$$\|\boldsymbol{v}\| = \sqrt{a^2 + b^2} \qquad \text{magnitude} \tag{4.7a}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \qquad \text{phase} \tag{4.7b}$$

where $\tan^{-1}$ is the arctangent function. The magnitude gives the length of the vector and the phase gives the orientation, as shown in Figure 4.2. This representation also gives us a geometrical picture of complex conjugation since

$$v^* = \|\boldsymbol{v}\| \cos\theta \hat{\boldsymbol{r}} - \|\boldsymbol{v}\| \sin\theta \hat{\boldsymbol{i}} \tag{4.8a}$$

can only be true if $\sin\theta \leq 0$ (and thus $\theta \in [0, -\pi]$) since $\|\boldsymbol{v}\| \geq 0$ and thus the effect of complex conjugation is a rotation in the opposite direction of θ. as shown in Figure 4.3

4.3 Euler's relation

One of the most important expressions involving complex numbers is *Euler's relation* (**?**):

$$e^{i\theta} = \cos\theta + i \sin\theta \tag{4.9}$$

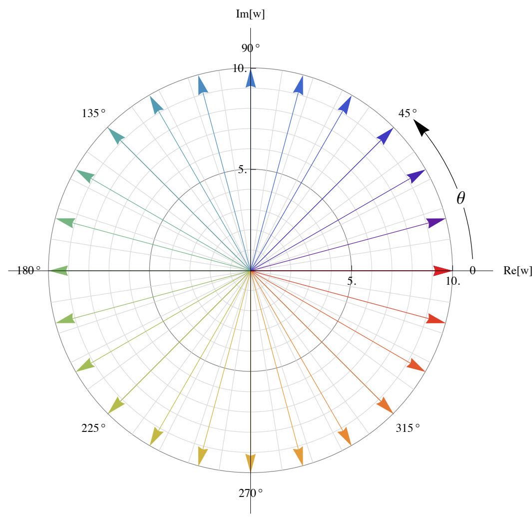


Figure 4.2 The complex plane: $w = re^{i\theta}$ where $r = 10$ and $\theta = \{0°, 15°, \ldots, 360°\}$. Colors increase from blue to red with increasing angle from the x axis:

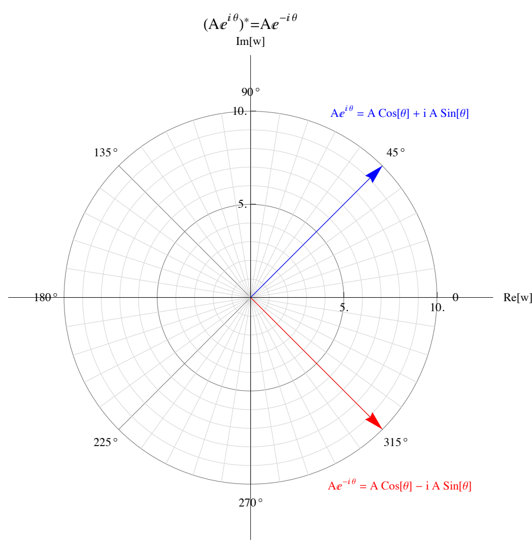


Figure 4.3 Complex conjugation. $w = re^{i\theta}$ and $w^* = re^{-i\theta}$.

Multiplying this expression by $r = \|\boldsymbol{v}\|$ gives the complex number

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta \tag{4.10}$$

which is exactly the expression Eqn 4.6 for a complex vector when plotted in the complex plane. We've chosen the letter r to emphasize that a vector fixed at the origin and moving only in the angular direction can be seen as the radius of a circle. Manipulation of complex numbers is much easier using this representation. Conjugation simply changes the sign in the exponent: $z^* = re^{-i\theta}$, and so we can check, for example, that the magnitude of z is indeed r:

$$\|z\| = \sqrt{z^* z} = \sqrt{\left(r^* e^{-i\theta}\right)\left(re^{i\theta}\right)} = r \tag{4.11}$$

because $r^* = r$ for a real number and the rule of multiplying exponentials with numbers (not matrices) in the exponent is:

$$e^a e^b = e^{a+b} \tag{4.12}$$

giving $e^{-i\theta}e^{i\theta} = e^0 = 1$.

The form Eqn 4.10 for complex numbers is ubiquitous in MRI as it provides a compact way to represent complex numbers in terms of the magnitude and phase, both of which play important roles in MRI. We will later find that the detected signal in MRI is defined by a vector that rotates in a plane (the so called *transverse plane*) about an axis defined by the main magnetic field (the *longitudinal axis*). The amplitude of the signal is associated with the magnitude of the vector, and the phase of the signal is associated with the phase of the vector. The MR signal is thus complex. For reasons that will become clear later, the reconstructed image is also complex. The "MR image" that one typically sees is actually the *magnitude* of the complex image. The *phase* of the image can have great practical importance in many applications. For example, in Figure 4.4 the magnitude of an image is shown along with the phase which represents the variations in the magnetic field that cause distortions, as will be discussed in Section **??**. Figure 4.4 clearly shows the advantage of representing complex numbers using the Euler representation. The magnitude and phase (Figure 4.4a and Figure 4.4b) have distinct physical meanings in MRI. Representing them as real and imaging components (Figure 4.4c and Figure 4.4d) is much harder to interpret because it has mixed these to components together.

4.4 The polarization decomposition

Here we introduce a somewhat more subtle trick that will prove quite useful in a few important situations. Notice that the real and imaginary components of a complex number $w = x + iy$ can be found from

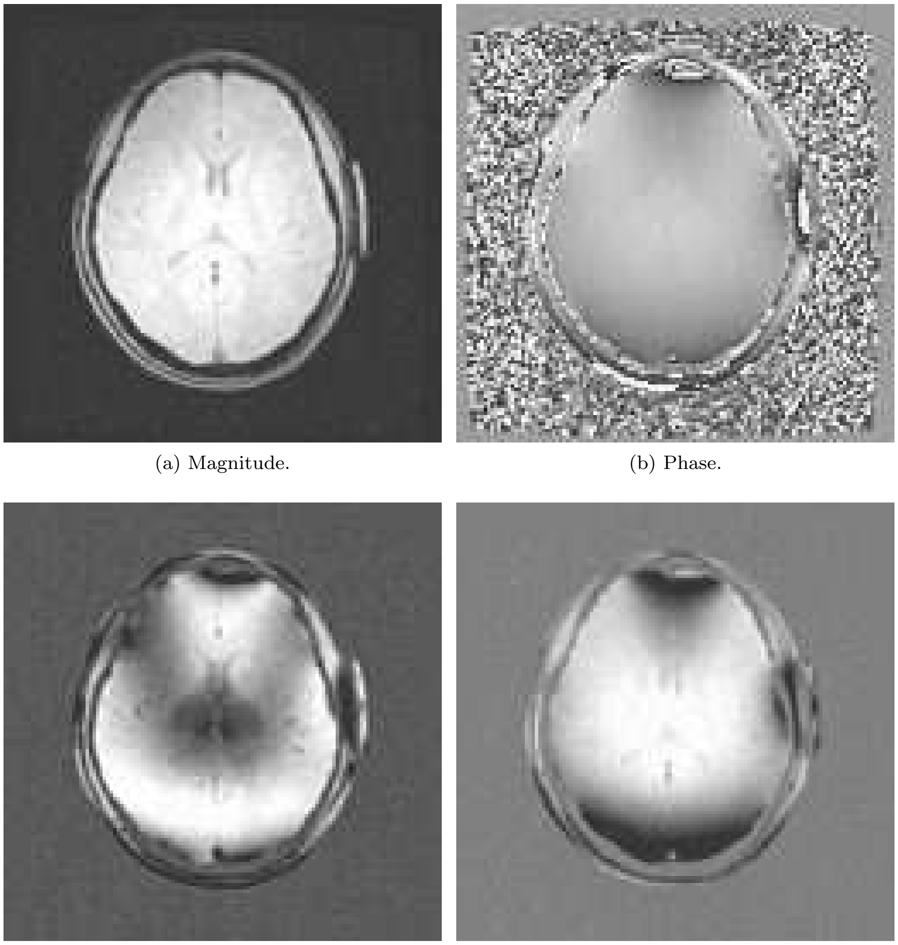
$$\Re[z] = \frac{1}{2}\left(z + z^*\right) \tag{4.13a}$$

$$\Im[z] = \frac{1}{2i}\left(z - z^*\right) \tag{4.13b}$$

Let's check this: $\Re[z] = \frac{1}{2}\left[(x+iy) + (x-iy)\right] = x$ and $\Im[z] = \frac{1}{2i}\left[(x+iy) - (x-iy)\right] = y$, so x and y are, as constructed, the real and imaginary parts of z. Let's now consider let θ be a function of time t so that $z(t) = re^{i\theta(t)}$ is a vector with length r rotating with phase $\theta(t)$. From Euler's relation, Eqn 4.9 can be written

$$r\cos\theta(t) = \frac{r}{2}\left[e^{i\theta(t)} + e^{-i\theta(t)}\right] \tag{4.14a}$$

$$r\sin\theta(t) = \frac{ir}{2}\left[e^{i\theta(t)} - e^{-i\theta(t)}\right] \tag{4.14b}$$



(a) Magnitude. (b) Phase.

(c) Real. (d) Imaginary

Figure 4.4 The MR signal and the reconstructed images are complex. The magnitude of the complex image data produces the image in (a) that is associated with the MR image typically presented. The phase image shown in (b) represents a magnetic field variations. The actual data are acquired as two ”channels” that are 90° out of phase: the *real* (c) and *imaginary* (d) channels. (**Replace this with an image where the phase has wrapping.**)

Thus the real component $r \cos\theta(t)$ can be described as the sum of two complex vectors of equal amplitude but *rotating in opposite* directions, i.e., with opposite *polarizations*. The imaginary component $r \sin\theta(t)$ can be described as the *difference* of two complex vectors of equal amplitude and opposite polarizations. This is shown in Figure 4.5

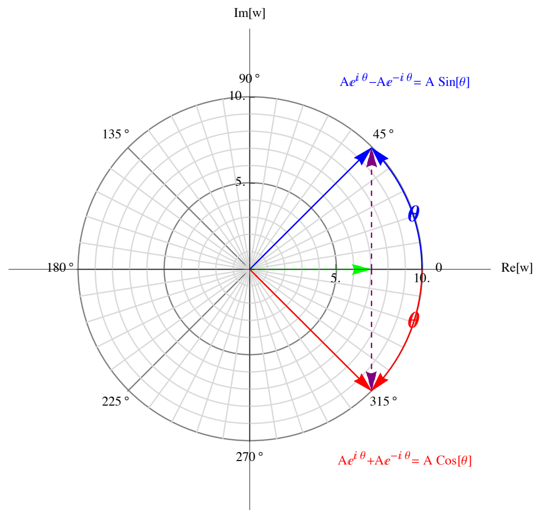


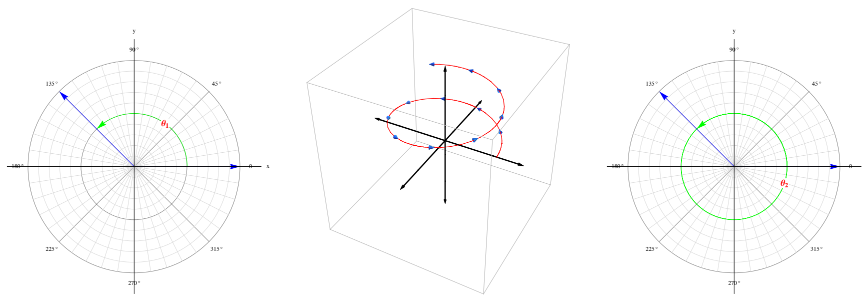
Figure 4.5 The polarization decomposition.

4.5 The ambiguity of phase

There are two interesting (and confounding) ambiguities that arises when attempting to determine the phase of a complex number. It is related to the fact that as a vector rotates around the complex plane, it retraces it path over and over.

First, consider the complex vector in that rotates through an angle $\theta = -3\pi/4$ as shown in Figure 4.6a. Let it then continue to rotate in the same direct through a full rotation (i.e., $\theta = 2\pi$. If we imagine this situation being a person climbing a spiral staircase, as in Figure 4.6c, where the height represents the total phase angle turned by the vector, the phase is unambiguous. However, in the complex plane the vector has arrived at the location $\theta = -3\pi/4$ again (Figure **??**). It is clear that this situation would be the same not matter how many additional 2π rotations are made, since that always brings the vector back to the same location on the circle. In other words, the phase $\theta = -3\pi/4$ is indistinguishable from the phase $\theta = -3\pi/4 + 2\pi n$ where n is an integer. Mathematicians state this as the phase being only determined *modulus* (or *mod*) 2π. This effects is easily seen using the Euler representation of complex numbers since $z = re^{i(\theta+2\pi n)} = re^{i\theta}e^{i2\pi n} = re^{i\theta}$ because of the identity $e^{i2\pi n} = 1$. This ambiguity is called *phase wrapping* and occurs in many contexts in MRI. An example is shown in Figure 4.8.

Now consider the situation shown in Figure 4.7 of two identical vectors starting from the same orientation ($\theta = 0$). The first vector rotates through and angle $\theta = -3\pi/4$ Figure 4.7a while the second vector rotates through and angle $\theta = +5\pi/4$ Figure 4.7b. Again we have the situation that these vectors are indistinguishable even though they have traversed different phase angles. In this case the ambiguity is called *aliasing*, since the vectors take on the appearance of one another. You've no doubt seen this effect in old Westerns where the sampling rate of the film is insufficient to capture the rotation of the spokes resulting in wagon wheels that appears to be rotating in the opposite direction. More pertinently, aliasing also occurs frequently in MRI

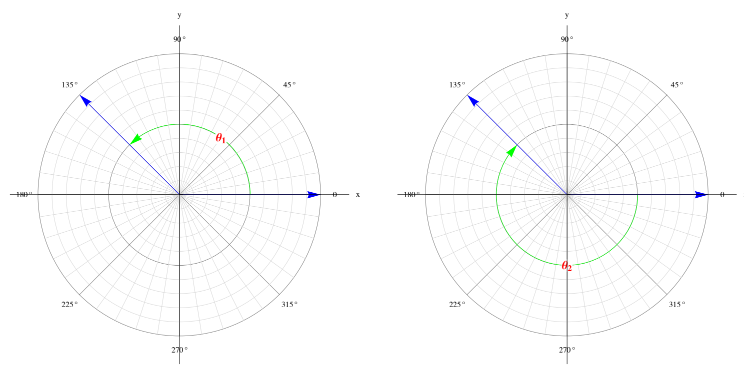


(a) $\theta = -3\pi/4$.

(b) $\theta = -3\pi/4 + 2\pi$ shown with time on the z axis shows that the phase angle increases continuously

(c) Phase of (b) ($\theta = -3\pi/4 + 2\pi$) in complex plane.

Figure 4.6 Phase wrapping. Rotation through an angle $\theta = -3\pi/4$ is indistinguishable from rotation through an angle $\theta = -3\pi/4 + 2\pi n$ where n is an integer.



(a) $\theta = -3\pi/4$.

(b) $\theta = +5\pi/4$.

Figure 4.7 Aliasing. Rotation through an angle $\theta = -3\pi/4$ is indistinguishable from rotation through an angle $\theta = +5\pi/4$.

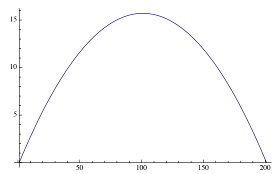
application, most notably in the apparent displacement in image intensities. This effect will be discuss in greater detail when we discuss MR image formation (Section **??**).

(**I was going to have an image aliasing example, but it seems confusing because it's the amplitude not the phase that is shown. We show this in mri chapter anyway.**)

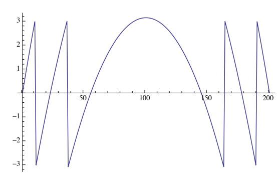
4.6 Vectors of complex numbers

Vectors can also be constructed with elements that are complex numbers. In this case the vector is said to be a *complex vector*. (Not to be confused with our original example of the vector representation of a single complex number in Figure 4.1).

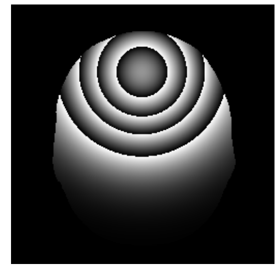
As we saw in Section 4.2 the magnitude of a complex number is found by multiplying it by its



(a) A quadratic phase with maximum value of 5π.



(b) The wrapped phase (a) has values between $-\pi \leq \theta \leq \pi$.



(c) Phase wrapping in image.

Figure 4.8 Phase wrapping in MRI. The display of phase jumps on a standard image grayscale plot transition from the top of the scale (white) to the bottom of the scale (black), and vise versa, producing images that appear to have rapidly varying intensities.

complex conjugate, then taking the square root. This can be readily extended to the vector case: in order for every element of a complex vector $\boldsymbol{v}$ to be multiplied by its complex conjugate, it is necessary to construct the complex vector whose elements are the complex conjugated elements of $\boldsymbol{v}$ and denote it by $\boldsymbol{v}^*$. To form the inner product from which we derive the length (Eqn **??**), the transpose of this vector is required: $\boldsymbol{v}^\dagger \equiv (\boldsymbol{v}^*)^t$. The complex conjugate transpose vector $\boldsymbol{v}^\dagger$ is called the *conjugate transpose* or *Hermitian conjugate* of $\boldsymbol{v}$ and the processes of transposition and conjugation have been wrapped into one symbol, †. The length of a complex vector is then just the square root of the inner product $\boldsymbol{v}^\dagger\boldsymbol{v}$ of the vector with its conjugate transpose vector:

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}^\dagger\boldsymbol{v}} \tag{4.15}$$

The inner product in $\mathbb{C}^N$ is the generalization of the dot product in $\mathbb{R}^N$.

In general, the inner product of two complex vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i=1}^{N} u_i^* v_i \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^N \qquad \text{standard basis} \tag{4.16}$$

where u_i, v_i are the components of $\boldsymbol{u}, \boldsymbol{v}$, respectively, in the standard basis, and where u^* is the complex conjugate of u. The *norm* of $\boldsymbol{u}$ is defined as

$$\|\boldsymbol{u}\| = \sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle} = \left(\sum_{i=1}^{N} |\boldsymbol{u}_i|^2 \right)^{\frac{1}{2}} \tag{4.17}$$

which is real and non-negative. In a basis other than the standand one, the inner product is

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i=1}^{N} w_i u_i^* v_i \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^N \qquad \text{standard basis} \tag{4.18}$$

where w_i are positive weights. In either case, the inner product and its norm satisfy the following properties for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{C}^N$ and $c \in \mathbb{C}$:

Properties of the Complex Inner Product

$$
\begin{aligned}
\text{Positivity :} \quad & \|\boldsymbol{u}\| > 0 \quad \text{for all} \quad \boldsymbol{u} \neq \mathbf{0} \\
\text{Hermiticity :} \quad & \langle \boldsymbol{u}, \boldsymbol{v} \rangle^* = \langle \boldsymbol{v}, \boldsymbol{u} \rangle \\
\text{Linearity :} \quad & \langle \boldsymbol{u}, a\boldsymbol{v} + b\boldsymbol{w} \rangle = a\langle \boldsymbol{u}, \boldsymbol{v} \rangle + b\langle \boldsymbol{u}, \boldsymbol{w} \rangle
\end{aligned}
$$

where $\mathbf{0}$ is the vector of all zeros for which $\|\mathbf{0}\| = 0$. As in the case of the dot product, two non-zero vectors for which $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$ are said to be *orthogonal*. Generally, the definition of the inner product depends upon the coordinate system used (one speaks of "the inner product with respect to the basis"). For our purposes, we shall be content with the Cartesian, or *standard* basis.